Sperner and Tucker's lemma

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Sperner's lemma

Brouwer's theorem

Let $\boldsymbol{B}$ be a $\boldsymbol{d}$-dimensional ball and $\boldsymbol{f}$ be a continuous $\boldsymbol{B} \longrightarrow \boldsymbol{B}$ map.

Then always

Theorem (Brouwer's theorem, 1911)

There exists $\boldsymbol{x} \in \boldsymbol{B}$ such that $\boldsymbol{f}(\boldsymbol{x}) = \boldsymbol{x}$.

Such an $\boldsymbol{x}$ is called a fixed-point.

Many applications (economics, game theory, algebra, geometry, combinatorics, ...).

Application

Assume that a tablecloth is on a table.

Take it.

Crumple it.

Throw it again on the table.

One point of the tablecloth is at the same position as before.

We know that such a fixed-point always exists, as soon as f is continuous.

But where is this fixed point ?

Nothing is said about the way of finding it in the statement of the theorem

So we can know that something exists without having seen it, and without knowing where we can find it.

The proofs

We can look at the proof.

Unfortunately, the first proofs were non-constructive.

Assuming that there is no fixed-point, we can build a new object – through continuous deformation (and then choice axiom) – whose existence is impossible.

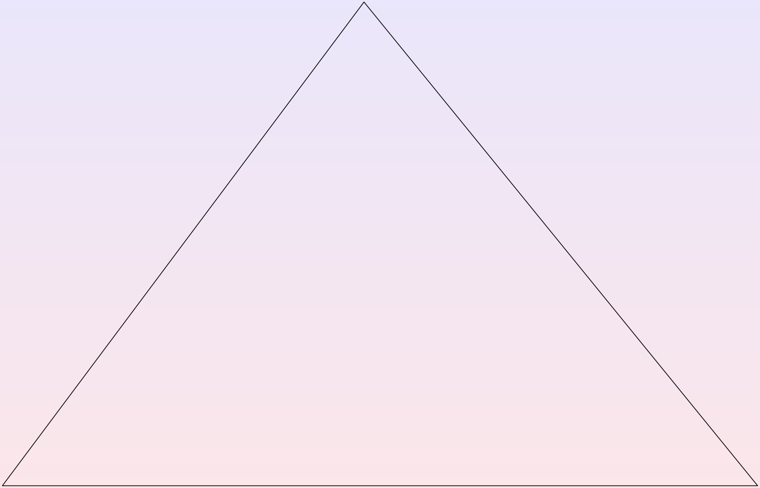
Sperner's lemma

In 1929, Sperner (a German mathematician) found a beautiful combinatorial counterpart of Brouwer's theorem, with a constructive proof.

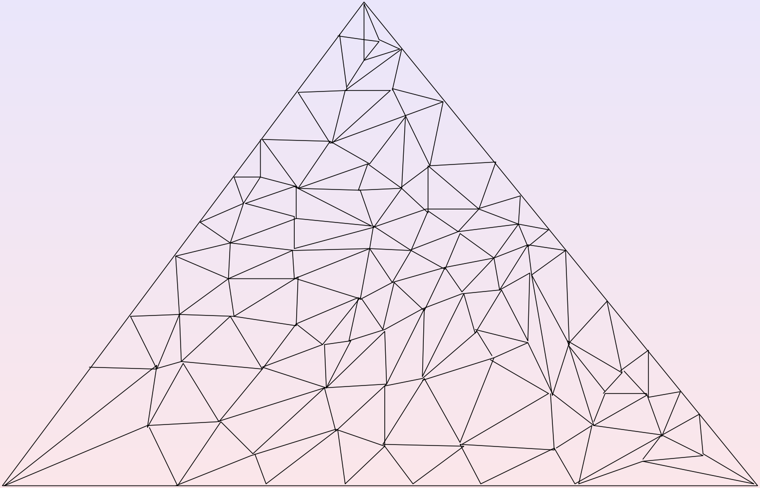
In dimension **2**: take a triangle T, whose vertices are respectively colored in blue, red and green. Assume that T is triangulated and denote by K this triangulation. Color the vertices of K in such a way that

- a vertex on a edge of T takes the color of one of the endpoints of the edge;
- a vertex inside T takes any of the three colors.

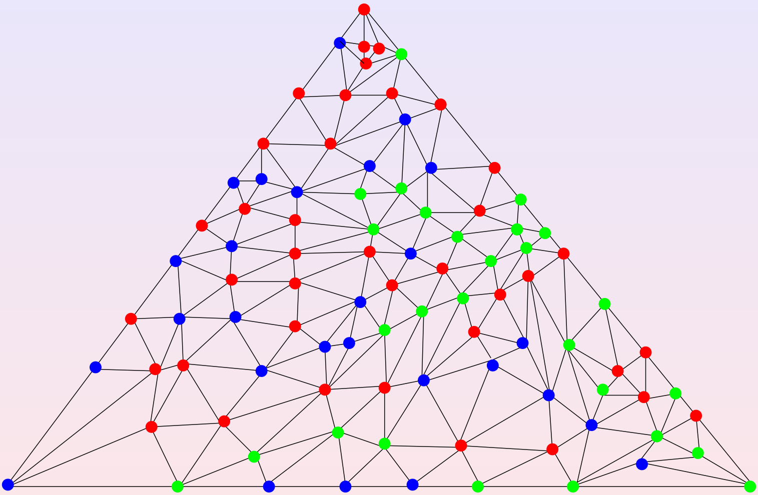
Then there is a fully-colored small triangle in K, that is, a small triangle of K having exactly one vertex of each color.



Sperner's lemma



Sperner’s lemma



Sperner's lemma

Lemma

Let K be a triangulation of a d-dimensional simplex T, whose vertices are denoted $v_0, v_1, \ldots, v_d$.

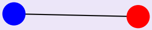
Let $\lambda : V(T) \rightarrow \{0, 1, \ldots, d\}$ such that $\lambda(v) = i$ implies that v_i is one of the vertices of the minimal face of T containing v.

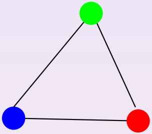
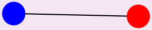
Then there is a small simplex $\sigma \in K$ such that $\lambda(\sigma) = \{0, 1, \ldots, d\}$.

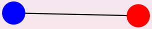
Such a λ is called a Sperner labelling. A simplex such that $\lambda(\sigma) = \{0, 1, \ldots, d\}$ is said to be fully-labelled.

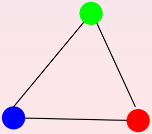
Proof

Proof in dimension 2

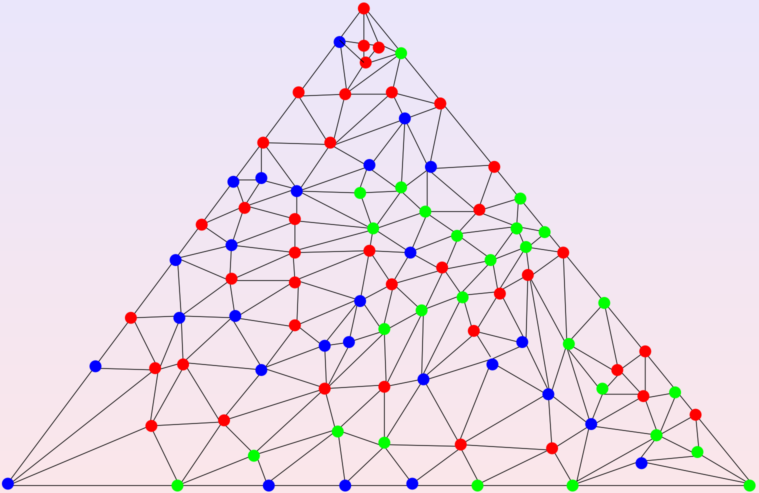
Start outside T. Go through the first .

Either you are in .
or their is another .

Repeat this operation. Since there is an odd number of  on the boundary of T, your walk will terminate inside T,

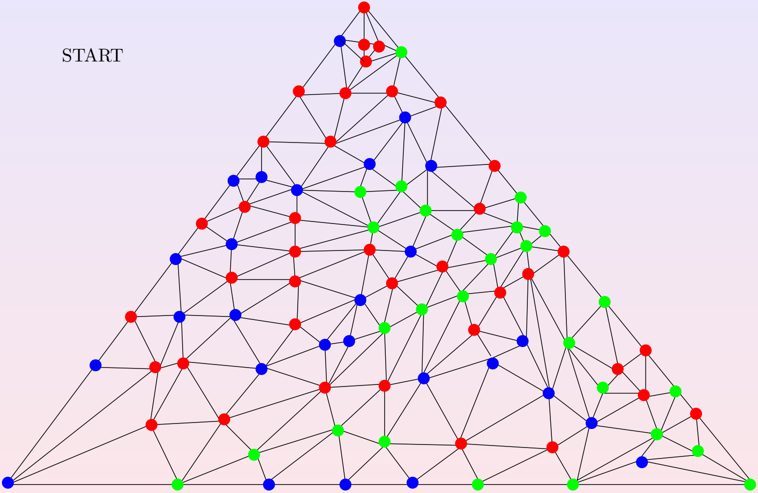
...necessarily in a .

A constructive proof



A constructive proof

START



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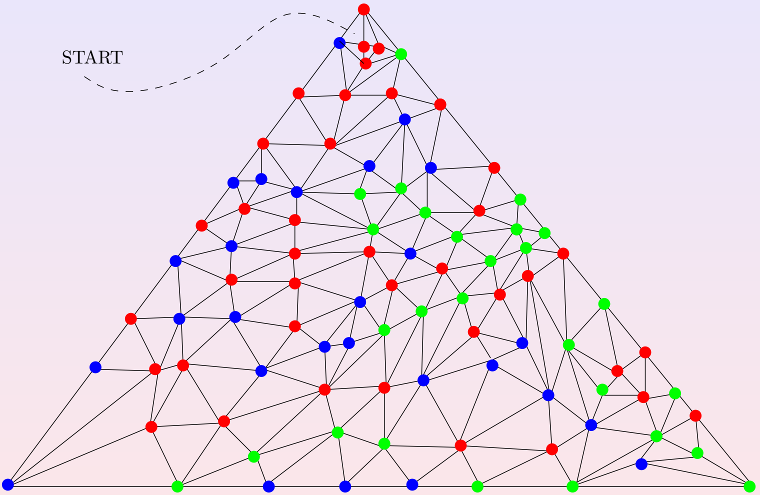

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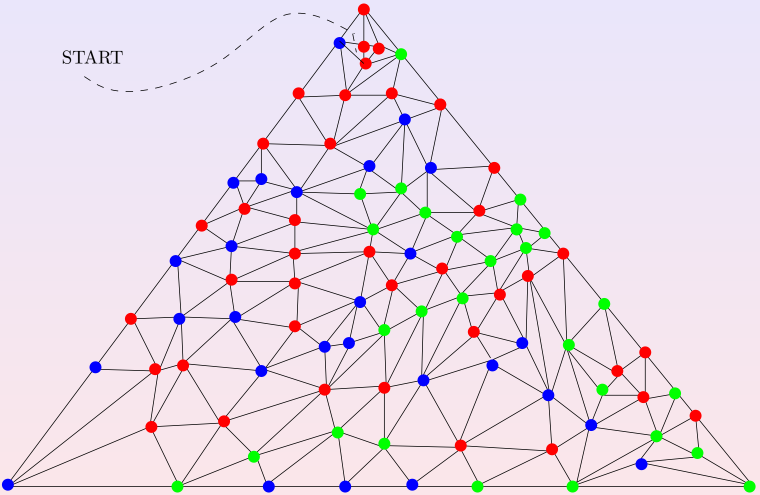

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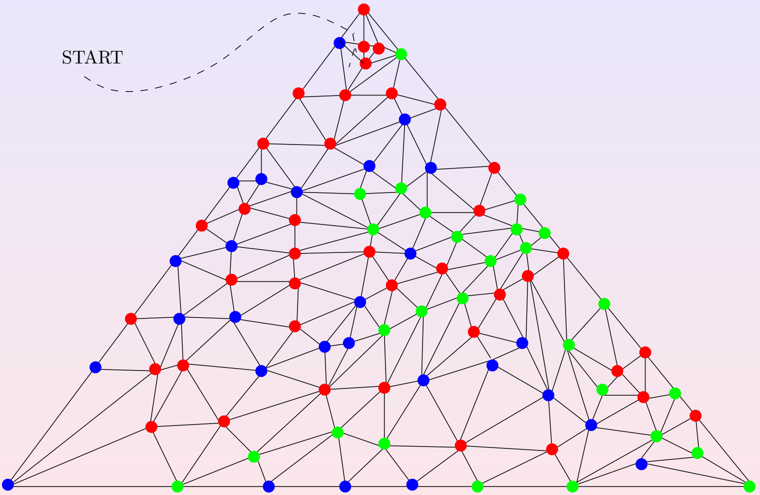

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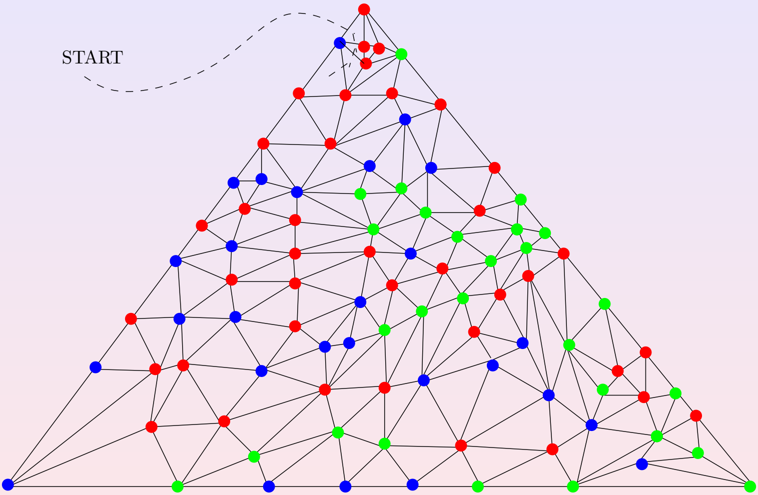

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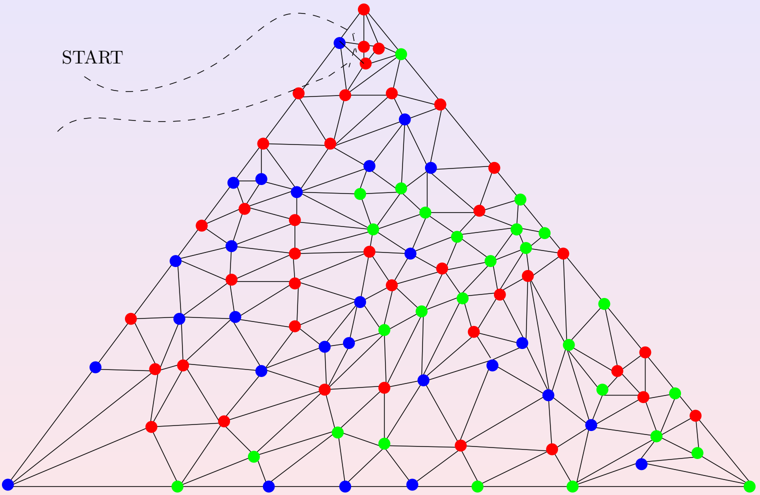

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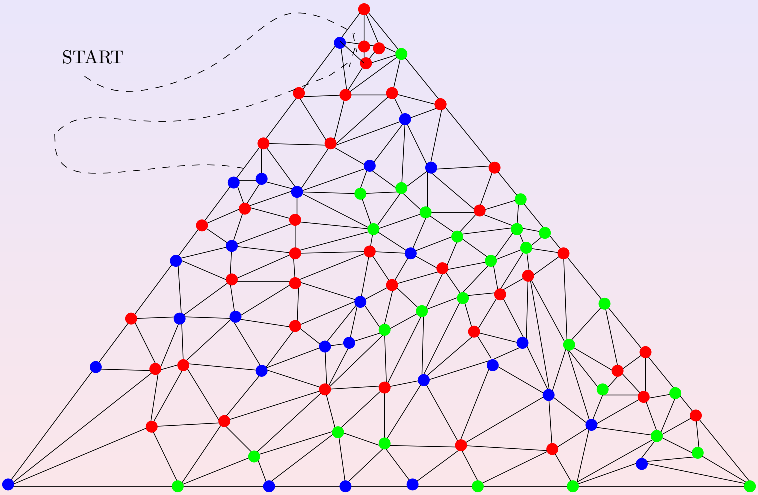

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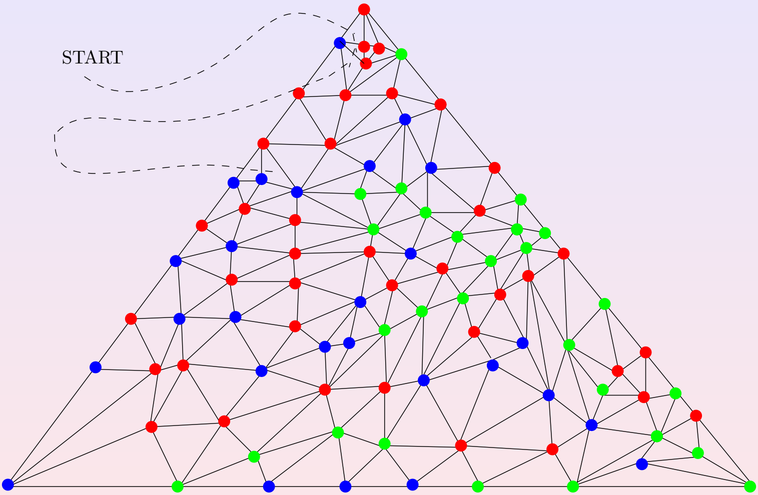

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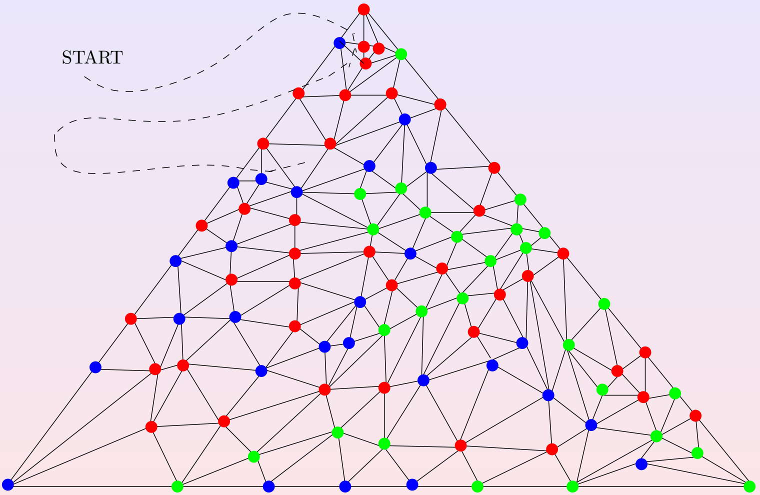

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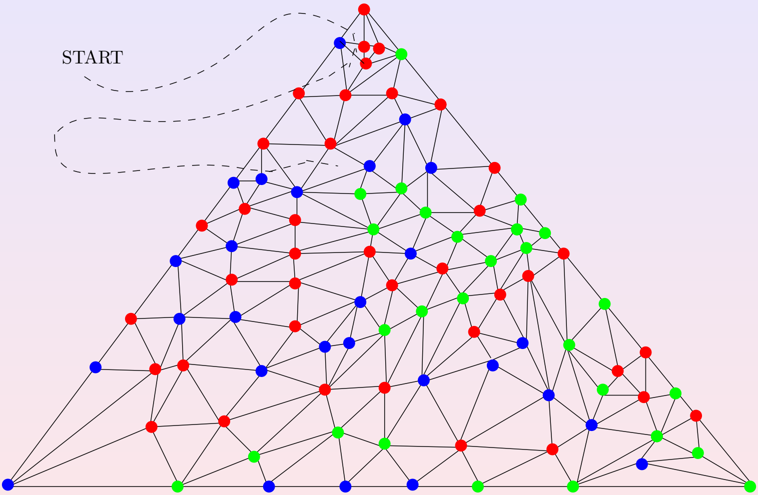

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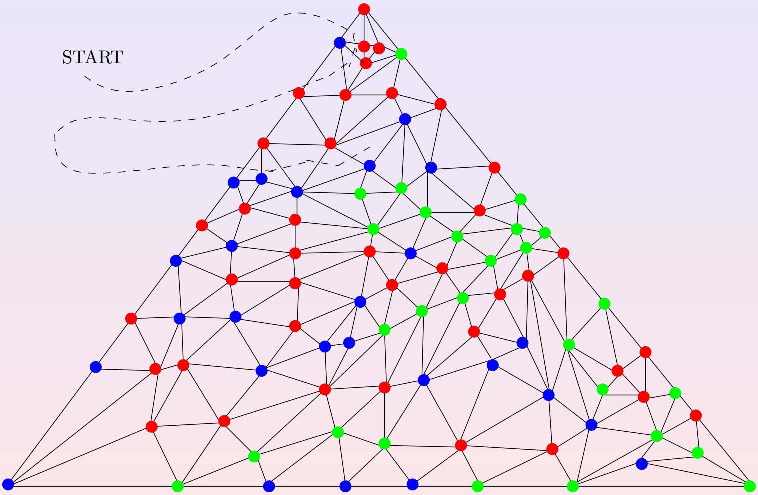

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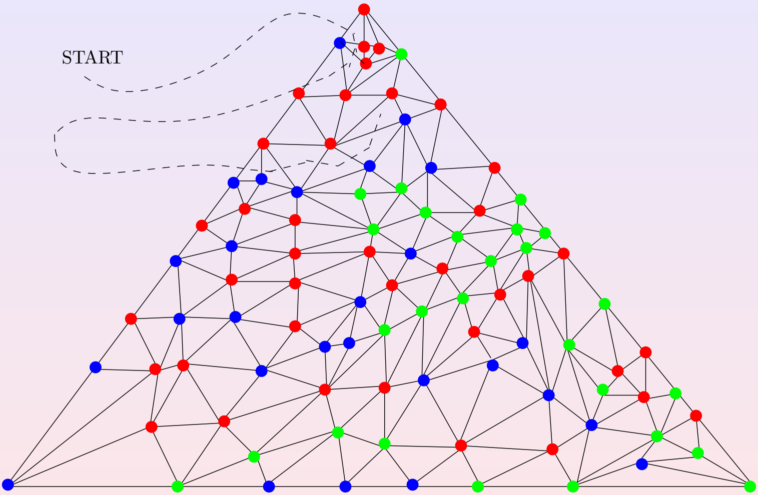

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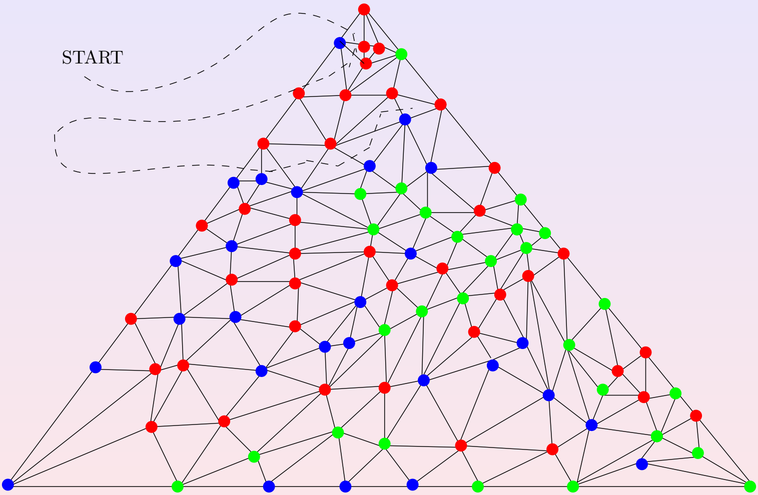

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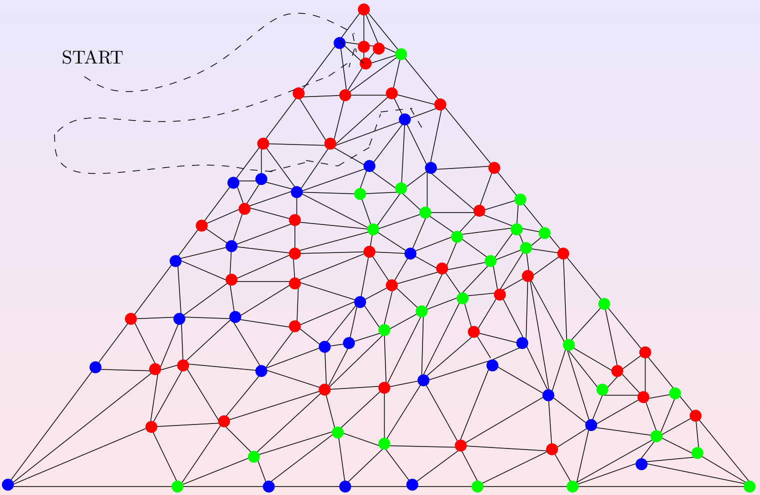

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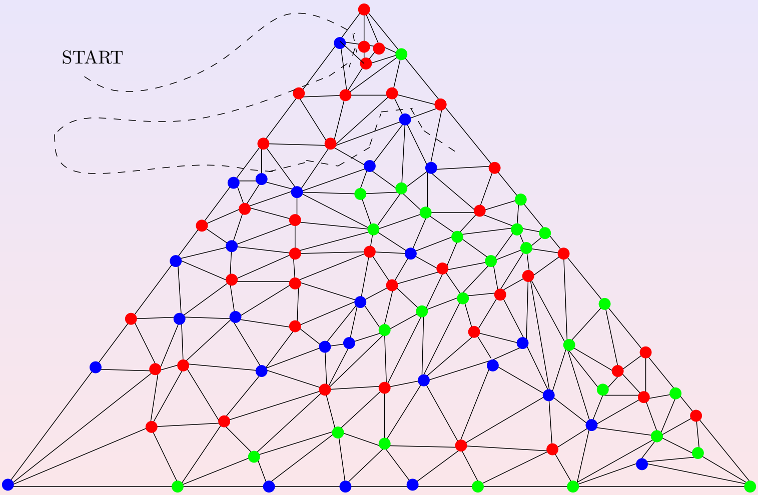

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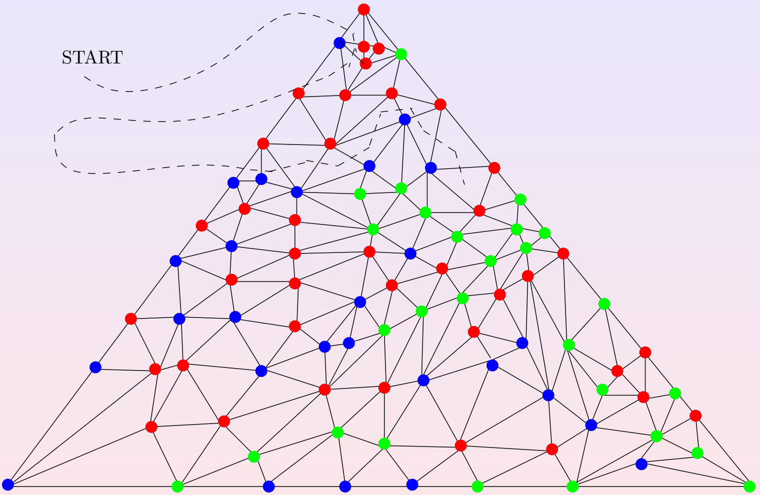

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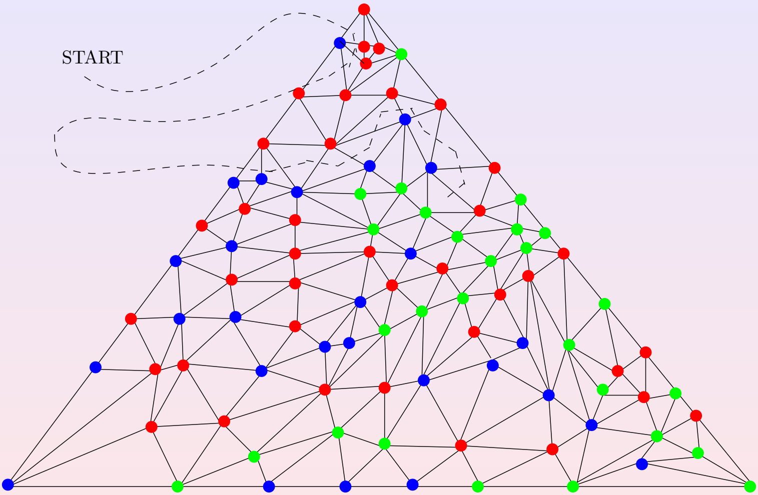

Dimension ≥ 2

Using induction, this proof can be used to prove Sperner's lemma in any dimension.

It provides an algorithm, which finds a fully-labelled simplex.

Oiks

Another proof proposed by Jack, using the oiks.

Lemma (Sperner's lemma bis)

Color the vertices of a triangulation of a d-sphere with $(d+1)$-colors. If there is a fully-colored simplex, then there is another one.

Oiks

A d-oik is a finite set V and a collection of $(d+1)$-subsets of V – called the rooms such that any d-subset of V is contained in an even number of rooms.

The triangulation K of the d-sphere is a d-oik. We denote by V its vertex set.

Another d-oik (Sperner): assume that all vertices of V are colored with $(d+1)$ colors. The subsets S such that $V \setminus S$ contains exactly one vertex of each color are the rooms of a d-oik.

Partition in rooms

Theorem

Let $C_1, \ldots, C_h$ be h oiks sharing a common vertex set. Assume that there is a partition of V in rooms $R_1, \ldots, R_h$ such that $R_i \in C_i$ for each $i = 1, \ldots, h$. Then there another partition of this kind.

Sperner's lemma is the consequence with $h = 2$ and with the two aformentioned d-oiks.

Back to Brouwer's theorem

Let f be a continuous map $T \to T$. Each point of T is identified by a triple (x_1, x_2, x_3) such that $x_1 + x_2 + x_3 = 1$ and $x_i \geq 0$ for each i. Write $f(x) = (f_1(x), f_2(x), f_3(x))$.

Take a triangulation K of mesh ϵ of T. Color each vertex of K with the smallest i such that $f_i(x) \leq x_i$. It provides a Sperner coloring. There is a small triangle τ_ϵ having exactly one vertex of each color. By compactness, it is possible to have a sequence of $\epsilon_k \to 0$ and a sequence τ_{ϵ_k} converging toward a point $x \in T$, which is necessarily a fixed-point of f.

We have seen that there is an algorithm that finds a fully-colored simplex. What about complexity ?

Papadimitriou has introduced in 1992 the PPAD class, which the class of search problems for which the existence is proved through the following argument

In any directed graph with one unbalanced node (node with outdegree different from its indegree), there must be another unbalanced node.

Moreover, he has proved that PPAD-complete problems exist, that is problems for which a polynomial algorithm would lead to a polynomial algorithm for any PPAD problem.

Fix n.

Consider the set of triple (n_1, n_2, n_3) of positive integers such that $n_1 + n_2 + n_3 = n$. Two triples (n_1, n_2, n_3) and (n'_1, n'_2, n'_3) are said to be neighbors if $\sum_{i=1}^{3} |n_i - n'_i| \leq 2$. It induces a triangulation of the 2-dimensional standard simplex.

Now assume that we have an oracle $\lambda(n_1, n_2, n_3)$ defined for all (n_1, n_2, n_3) giving a Sperner labelling on the aformentioned triangulation.

Is it possible to find a fully-labelled simplex in polynomial time ?

Theorem (Chen Deng 2009)

Sperner is PPAD-complete in dimension $\mathbf{2}$.

For dimension $\mathbf{3}$, it was already proved by Papadimitriou.

Another Sperner's lemma

Theorem (De Loera, Prescott, Su, 2003)

Let K be a triangulation of a d-dimensional polytope P, whose vertices are denoted $v_0, v_1, \ldots, v_n$.

Let $\lambda : V(P) \to \{0, 1, \ldots, n\}$ such that $\lambda(v) = i$ implies that v_i is one of the vertices of the minimal face of P containing v.

Then there is at least $n - d$ small simplices $\sigma \in K$ such that $|\lambda(\sigma)| = d + 1$.

Remark. [M.] $n - d$ can be improved to $n - d + \max_{v \in V} \deg(v)/d$.

Another Sperner's lemma

Theorem (Babson, 2007)

Let K be a triangulation of a d-dimensional simplex T. Let n_1, n_2 be positive integers such that $n_1 + n_2 = d + 2$. Let λ_1, λ_2 be two Sperner labellings. There there exists a small simplex $\sigma \in K$ such that $|\lambda_1(\sigma)| \geq n_1$ and $|\lambda_2(\sigma)| \geq n_2$.

Usual Sperner: $n_1 = d + 1$ and $n_2 = 1$.

Even for $d = n_1 = n_2 = 2$, it is not at all easy.

No constructive proof known !

Another Sperner's lemma

Theorem (Babson, 2007)

Let K be a triangulation of a d-dimensional simplex T. Let $n_1, \ldots, n_q$ be positive integers such that $\sum_{i=1}^{q} n_i = d + q$. Let $\lambda_1, \ldots, \lambda_q$ be q Sperner labellings. There there exists a small simplex $\sigma \in K$ such that $|\lambda_i(\sigma)| \geq n_i$ for all $i = 1, \ldots, q$.

No constructive proof known !

Tucker's lemma

The Borsuk-Ulam theorem

Let $\boldsymbol{B}$ be a $\boldsymbol{d}$-dimensional ball and $\boldsymbol{f}$ be a continuous $\boldsymbol{B} \longrightarrow \mathbb{R}^{\boldsymbol{d}}$ map such that for each $\boldsymbol{x} \in \partial \boldsymbol{B}$, we have $\boldsymbol{f}(-\boldsymbol{x}) = -\boldsymbol{f}(\boldsymbol{x})$.

Then always

Theorem (Borsuk-Ulam, 1933)

There exists $\boldsymbol{x} \in \boldsymbol{B}$ *such that* $\boldsymbol{f}(\boldsymbol{x}) = \mathbf{0}$.

Many applications (algebra, discrete geometry, combinatorics, analysis...).

Application

Take a sandwich with ham, tomato and bread.

There is a plan that divides the sandwich in two parts, each of them having the same quantity of ham, tomato and bread.

Existence

The same problem as for Brouwer's theorem arises: where and how can we find the x such that $f(x) = 0$?

The first proofs were completely non-constructive.

Tucker's lemma

Tucker found a combinatorial counterpart of the B-U theorem, with a constructive proof.

Lemma (Tucker, 1946)

Assume that the $\boldsymbol{d}$-ball $\boldsymbol{B}$ is triangulated, with a triangulation $\boldsymbol{K}$ that induces a centrally symmetric triangulation on $\partial\boldsymbol{B}$. Let λ be a labelling of the vertices with $\{-1,+1,\ldots,-\boldsymbol{d},+\boldsymbol{d}\}$, and assume that $\lambda(-\boldsymbol{v}) = -\lambda(\boldsymbol{v})$ for any $\boldsymbol{v} \in \partial\boldsymbol{B}$. Then there exists an edge whose endpoints have opposite labels.

Proof in dimension 2

Start outside ***B***. Go through the first 

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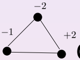
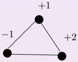
Either you are in 

or a 

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or their is another 

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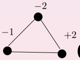
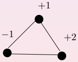
Repeat this operation. Since there is an odd number of 

on the boundary of ***T***, your walk will terminate inside ***B***,

...necessarily in a 

or a 

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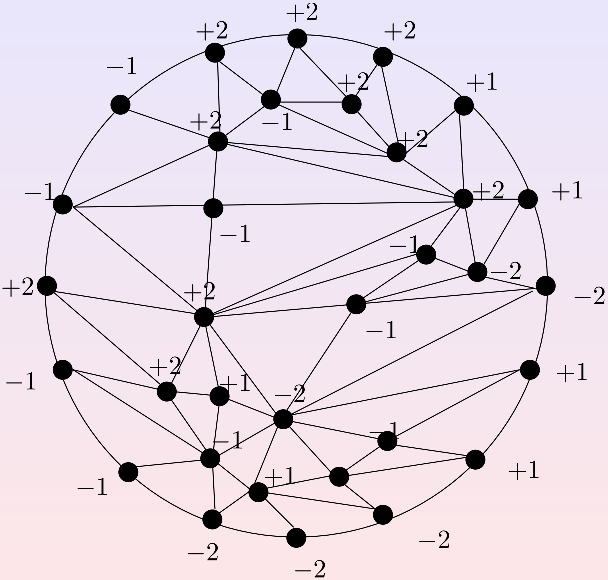
A constructive proof



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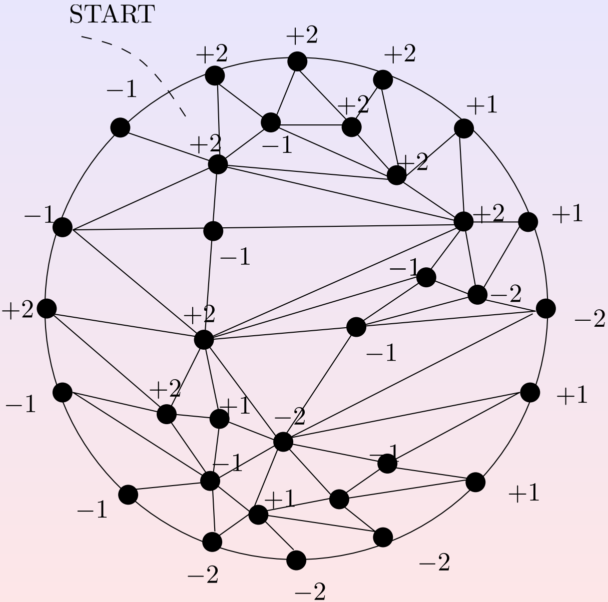

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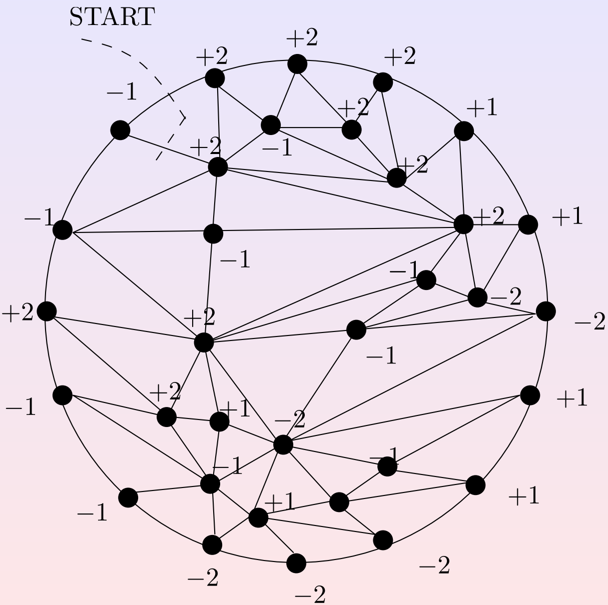

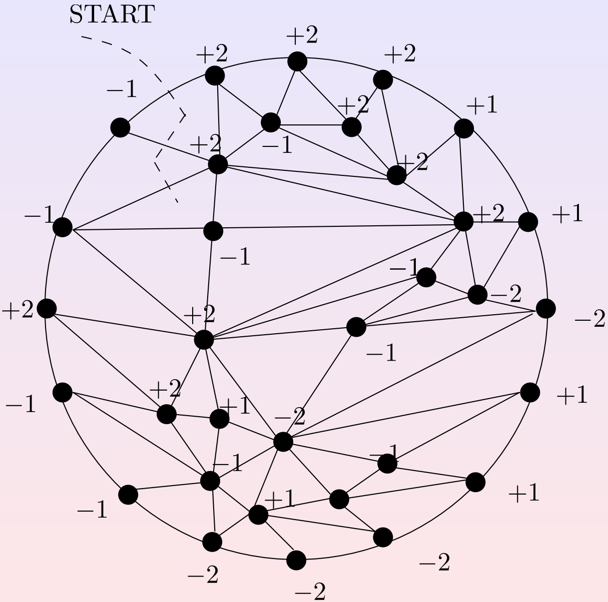
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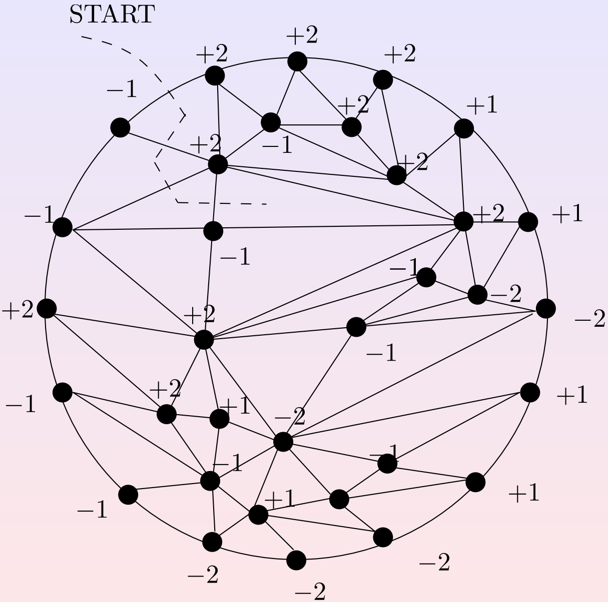
A constructive proof



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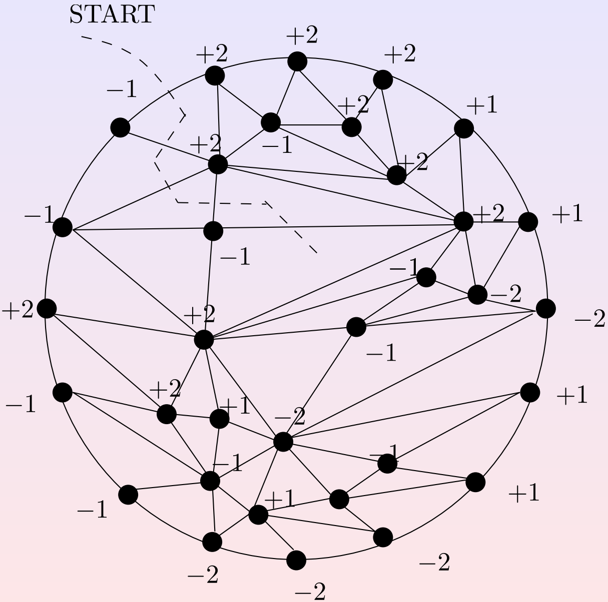

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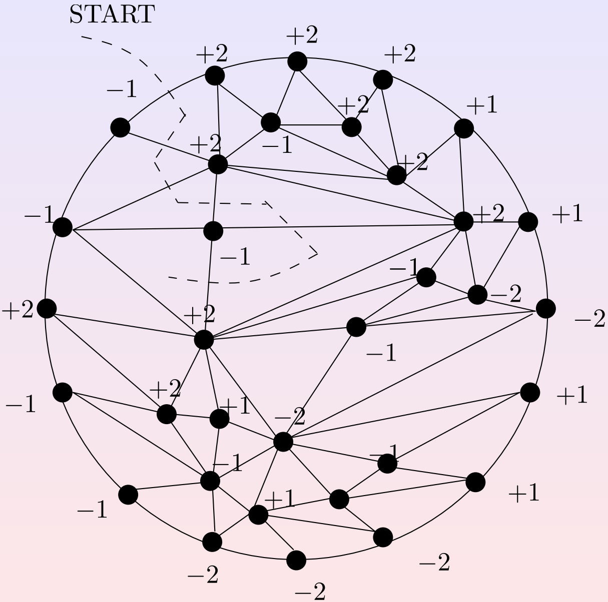

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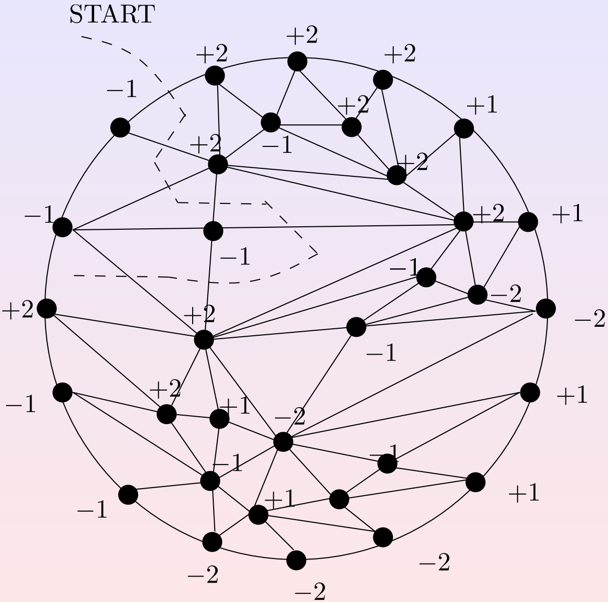

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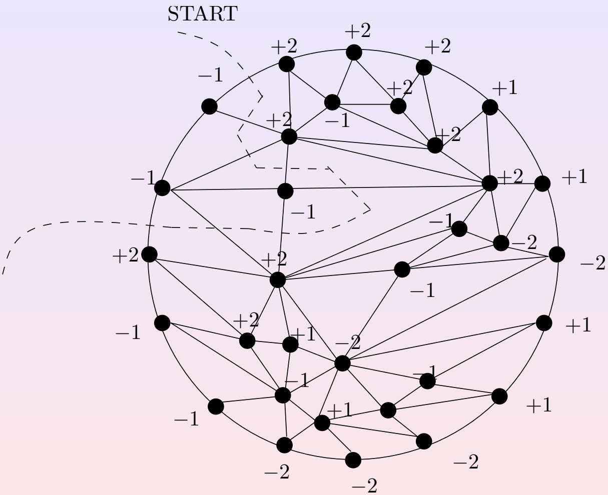

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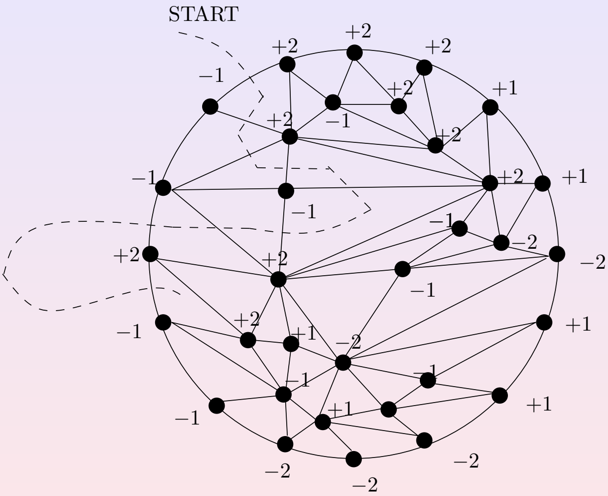

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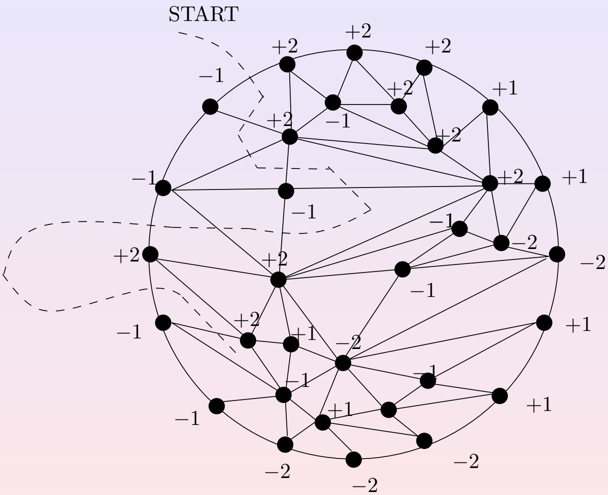

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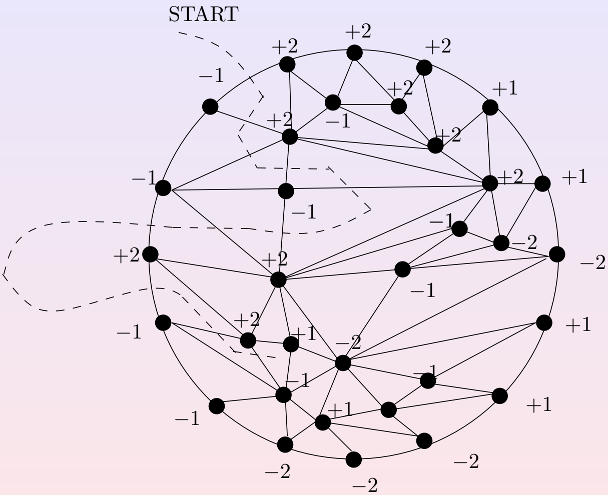

Dimension ≥ 2

Using induction, this proof can be used to prove Tucker's lemma in any dimension.

It provides an algorithm, which finds an antipodal edge.

More details

If we want to be precise in the proof, the right thing to prove is

Lemma (Ky Fan, 1952)

Assume that the d-ball B is triangulated, with a triangulation K that induces a centrally symmetric triangulation on ∂B. Let λ be a labelling of the vertices with $\{-1, +1, \ldots, -m, +m\}$, and assume that $\lambda(-v) = -\lambda(v)$ for any $v \in \partial B$ and that there exists no edge whose endpoints have opposite labels.
Then there is an odd number of d-dimensional alternating simplices.

A k-dimensional alternating simplex is a simplex whose labels are of the form $-j_1, +j_2, \ldots, (-1)^k j_k$ or $+j_1, -j_2, \ldots, (-1)^{k-1} j_k$ with $j_1 < j_2 < \ldots < j_k$.

Constructive proof

By induction, we know that there is an odd number of $(d-1)$-dimensional alternating simplices on ∂B.

Go through such a simplex, you enter a d-dimensional simplex either having another facet being a $(d-1)$-dimensional alternating simplex, or being itself a d-dimensional alternating simplex.

And go on.

Back to the Borsuk-Ulam theorem

Let K be a triangulation of the d-ball B with mesh ϵ and f be the continuous map $B \longrightarrow \mathbb{R}^d$ being antipodal on the boundary of B.

Label the vertex v with the smallest i such that $|f_i(x)| = \max_{j=1,\ldots,d} |f_j(x)|$, and put the sign of $f_i(x)$ as sign of the label. This labelling satsfies the requirement of Tucker's lemma: there is an antipodal edge $(-i, +i)$.
When $\epsilon \longrightarrow 0$, we can find a converging sequence of antipodal edges (by compactness), which means that there is a sequence (x_n) such that $\lim_{n\to\infty} \max_{j=1,\ldots,d} |f_j(x_n)| = 0$.

Theorem (Pálvölgyi 2009)

Tucker is PPAD-complete in dimension $\mathbf{2}$.