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Note

Cycles in Graphs with Prescribed Stability Number and Connectivity

MEKKIA KOUIDER

L.R.I., U.A. 410 C.N.R.S., Université Paris-Sud, Orsay, France

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D. Amar, I. Fournier, A. Germa, R. Häggkvist, and C. Thomassen conjectured that the vertices of a k-connected graph with stability number α can be covered by $\lceil \alpha/k \rceil$ cycles. We prove this conjecture. © 1994 Academic Press, Inc.

Bondy [BO] observed that every graph with stability number α admits a partition of its vertex set into at most α cycles, edges, and vertices. Here, we consider k-connected graphs, where $k \geqslant 2$. In this case, since each vertex lies in a cycle, it is natural to consider coverings of the vertices by cycles alone. Chvátal and Erdős [CE] proved that if G is a k-connected graph with stability number α, where $\alpha \leqslant k$, then G is hamiltonian. In 1982, D. Amar, I. Fournier, A. Germa, R. Häggkvist, and C. Thomassen (see [FO]) proposed a generalization of this theorem. We verify their conjecture by proving

THEOREM 1. *If G is a k-connected graph with stability number α, the vertices of G can be covered by $\lceil \alpha/k \rceil$ cycles.*

The bound is sharp because, for every integer $\alpha \geqslant k$, the complete bipartite graph $K_{k,\alpha}$ is k-connected, has stability number α and cannot be covered by fewer than $\lceil \alpha/k \rceil$ cycles. The case $k=2$ was settled by C. Thomassen (see [FO]), the cases $\alpha = k+1$ and $\alpha = k+2$ by Amar, Fournier, Germa, and Häggkvist [AFGH]. The case $k=3$ was studied in [MA]. We shall prove the conjecture in the following stronger form.

THEOREM 2. *If H is a subgroph of a k-connected graph G, then either the vertices of H can be covered by one cycle of G or there exists a cycle C of G such that*

$$\alpha(H \setminus C) \leqslant \alpha(H) - k.$$

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Indeed, to see that Theorem 2 implies Theorem 1, set $H_1 := G$ and $H_{i+1} := H_i \backslash C_i$ for $i = 1, ..., r$, where C_i is the cycle C whose existence is guaranteed by Theorem 2 when $H := H_i$, and r is the greatest integer such that $H_r \neq \emptyset$. The vertices of G are covered by the cycles C_i, $1 \leqslant i \leqslant r$. Moreover, $r \leqslant \lceil \alpha/k \rceil$ since $\alpha(H_{i+1}) \leqslant \alpha(H_i) - k$, $1 \leqslant i \leqslant r$.

Proof of Theorem 2. For a subgraph X of G, set $\alpha_H(X) := \alpha(X \cap H)$. Suppose that there is no cycle C of G such that $H \backslash C = \emptyset$. For each cycle C, let F_C be a smallest component of $G \backslash C$ such that $F_C \cap H \neq \emptyset$, and choose a cycle C for which

(i) $\alpha_H(G \backslash C) = \alpha(H \backslash C)$ is as small as possible;

(ii) subject to (i), $F := F_C$ is as small as possible.

Set $W := G \backslash (C \cup F)$.

With respect to a specific orientation of C, let $c_1, c_2, ..., c_m$ be the neighbours of F, in this order, on C, and set $C_i := C(c_i, c_{i+1})$, $1 \leqslant i \leqslant m$, where indices are taken modulo m. Note that $m \geqslant k$ as G is k-connected.

Claim 1. *Let C' be a cycle of G which contains $c_1, c_2, ..., c_m$ and at least one vertex of F. Set $F' := F \backslash C'$ and $W' := G \backslash (C' \cup F')$. Then*

$$\alpha_H(W') > \alpha_H(W).$$

Proof of Claim 1. If $F' \cap H \neq \emptyset$, $G \backslash C'$ has a component F'' and $F'' \subset F$ such that $F'' \cap H \neq \emptyset$. By the choice of C, $\alpha_H(G \backslash C') > \alpha_H(G \backslash C)$, so

$$\alpha_H(W') = \alpha_H(G \backslash C') - \alpha_H(F') > \alpha_H(G \backslash C) - \alpha_H(F) = \alpha_H(W).$$

If $F' \cap H = \emptyset$,

$$\alpha_H(W') = \alpha_H(G \backslash C') \geqslant \alpha_H(G \backslash C) > \alpha_H(W).$$

Thus, in either case, $\alpha_H(W') > \alpha_H(W)$.

For $i \neq j$, there is a C-bypass P_{ij} connecting c_i and c_j whose internal vertices belong to F, and hence a cycle $C_{ij} := P_{ij} C[c_j, c_i]$ such that $C_{ij} \cap F \neq \emptyset$. Setting $C' := C_{i,i+1}$ in Claim 1, we obtain

$$\alpha_H(C_i \cup W) > \alpha_H(W), \qquad 1 \leqslant i \leqslant m. \tag{*}$$

We remark that C_i is not empty. ▌

Claim 2. *Let $V_i \subseteq C_i$, $1 \leqslant i \leqslant m$, and suppose that there is not path internally disjoint from $C \cup F$ with its ends in different members of* *the set* $\{V_1, V_2, ..., V_m\}$. *Denote by* F_i *the union of the components of* W *whose neighbours lie in* V_i, $1 \leqslant i \leqslant m$, *and set* $F_0 := W \backslash (\bigcup_{i=1}^{m} F_i \cup F)$. *Then*

$$\alpha_H\left(\bigcup_{i=1}^{m} V_i \cup W\right) = \alpha_H(F_0) + \sum_{i=1}^{m} \alpha_H(V_i \cup F_i).$$

Proof of Claim 2. By the hypothesis on the V_i, $F_i \cap F_j = \emptyset$, $1 \leqslant i < j \leqslant m$. Thus $\{F_0, V_1 \cup F_1, ..., V_m \cup F_m\}$ is a partition of $\bigcup_{i=1}^{m} V_i \cup W$. Moreover, again by the hypothesis on the V_i, no edge of G has its ends in different members of this partition, and the claim follows.

Set $D_i := C(c_i, d_i]$, $1 \leqslant i \leqslant m$, where d_i is the first vertex of C_i such that

$$\alpha_H(D_i \cup W) > \alpha_H(W). \qquad (**)$$

Note that d_i is well-defined, by (*). ▮

CLAIM 3. *There is no path internally disjoint from* $C \cup F$ *with its ends in different members of the set* $\{D_1, D_2, ..., D_m\}$.

Proof of Claim 3. Suppose, to the contrary, that there are vertices $v_r \in V(D_r)$ and $v_s \in V(D_s)$ that are connected by a path Q_{rs} internally disjoint from $C \cup F$. Set $V_i := C(c_i, v_i)$, $i = r, s$, and $V_i := \emptyset$, $i \neq r, s$, and assume that v_r and v_s are chosen so as to minimize the sum of the lengths of V_r and V_s. Applying Claim 2 and noting that (by the definition of d_i when $i = r, s$, and by the definition of V_i when $i \neq r, s$) $\alpha_H(V_i \cup W) = \alpha_H(W)$ for all i and hence $\alpha_H(V_i \cup F_i) = \alpha_H(F_i)$ for all i, we obtain

$$\alpha_H(V_r \cup V_s \cup W) = \sum_{i=0}^{m} \alpha_H(F_i) = \alpha_H(W).$$

On the other hand, setting $C' := Q_{rs} C[v_s, c_r] P_{rs} \bar{C}[c_s, v_r]$ in Claim 1, we have

$$\alpha_H(V_r \cup V_s \cup W) > \alpha_H(W).$$

This contradiction establishes Claim 3. ▮

Theorem 2 now follows by setting $V_i := D_i$, $1 \leqslant i \leqslant m$, in Claim 2 (which we may do, by Claim 3), and noting that $\alpha_H(D_i \cup F_i) > \alpha_H(F_i)$, $1 \leqslant i \leqslant m$, by (**):

$$\begin{aligned} \alpha(H) = \alpha_H(G) &\geqslant \alpha_H\left(\bigcup_{i=1}^{m} D_i \cup W\right) + \alpha_H(F) \\ &= \alpha_H(F_0) + \sum_{i=1}^{m} \alpha_H(D_i \cup F_i) + \alpha_H(F) \\ &\geqslant \alpha_H(G \backslash C) + m \geqslant \alpha(H \backslash C) + k. \end{aligned}$$

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