Redundancy and Helly

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The classical Helly's Theorem about finite sets of convex sets is given an unusually simple proof based on a 'Redundancy Lemma'. Because the proof is topological it extends immediately to a Helly's Theorem for the well-known combinatorial topology representation of oriented matroids which is reviewed. The same proof is then used to strengthen Helly's Theorem in a useful way relative to the Farkas Lemma, both for linear inequality systems and for topologically represented oriented matroids.

Helly's Theorem states that:

(1) *For any finite set S of convex subsets of a d-dimensional Euclidean space E, either* $\cap(S) \neq \Phi$ *or there is a subset* $R \subseteq S$*, of size at most* $d + 1$*, such that* $\cap(R) = \Phi$*.*

By (closed) half-space $H^{\leq}$ of the space E we mean the solution set of some single linear inequality $a_i z \leq b_i$ in the d variable row-vectors z of a Cartesian coordinatization of the Euclidean space E. Or more generally where E is a subspace of a space E' (i.e., where E is the solution set of a set of linear equations in variables coordinatizing space E'), by half-space of E we mean $H^{\leq} \cap E$ where $H^{\leq}$ is a half-space of E'. The empty set and E itself are half-spaces of E. The other half-spaces of E, but not Φ nor E, are called proper half-spaces of E. For any proper half-space $H^{\leq}$ of E, $H^{=}$ denotes the boundary of $H^{\leq}$ (a 'hyperplane' of E), that is, the solution set of $a_i z = b_i$, where $H^{\leq}$ is the solution set of $a_i z \leq b_i$. $H^{<}$ denotes the 'open half-space' $H^{\leq} - H^{=}$. $H^{\geq}$ denotes the companion (closed) half-space $E - H^{<}$, and $H^{>}$ denotes the companion open half-space $E - H^{\leq}$.

A (convex) polyhedron in E means the intersection of a finite set of half-spaces of E. The dimension of the empty set is -1; the dimension of a single point is 0; and so forth as usual. Of course subspaces, half-spaces, and polyhedra, are convex sets, and so an important special case of Helly's Theorem is:

(2) *For any finite set S of half-spaces of a d-dimensional Euclidean space E, either* $\cap(S) \neq \Phi$ *or there is a subset* $R \subseteq S$*, of size at most* $d + 1$*, such that* $\cap(R) = \Phi$*.*

In other words:

(2') A finite set $Az \leq b$ of linear inequalities in the d variables $z = (z_1, \ldots, z_d)$ has a solution z or else there is a subset $A'z \leq b'$ of at most $d + 1$ of the inequalities such that $A'z \leq b'$ has no solution.

Helly's Theorem (1) where S is a finite set of polyhedra follows immediately from (2) since each polyhedron is the intersection of a finite set of half-spaces. Helly's Theorem (1) for a finite set S of general convex sets follows from the theorem for a finite set of polyhedra: assuming that the intersection of each size $d + 1$ subset of S is non-empty, choose one point from each of these intersections to obtain a finite set, say P. Let the members of S' be the convex hulls of the finite sets $P \cap C$, for the members C of S. Assuming the theorem that the convex hull of any finite set of points in E is a polyhedron, we have (1) for S', which implies (1) for S.

My purpose here is to give an unusually simple proof of (2). From it we will see a way in which Helly's Theorem naturally generalizes to topology and to oriented matroids. Michel Las Vergnas invited me to write this article because he remarked that there has been puzzlement about how to extend Helly's Theorem to oriented matroids. In that regard, see p. 382 of [2].

(3) The Redundancy Lemma. *Let $H^{\leq}$, having boundary $H^{=}$, be a proper half-space of E. Let R' be a set of half-spaces of E such that $[\cap(R')] \cap H^{=} = \Phi$. Then either*:

(a) *$R = R' \cup \{H^{\leq}\}$ is such that $\cap(R) = \Phi$ or else*
(b) *$H^{\leq}$ is 'redundant' with R'. That is: $\cap(R') \subseteq H^{\leq} - H^{=} = H^{<}$, and so $\cap(R') = [\cap(R')] \cap H^{<}$.*

PROOF OF HELLY (2) FROM (3). (2) is clear if some member of S is empty, or if every member of S is E. If $d = 0$ this must be the case since then E is a single point. Otherwise, if $H^{\leq}$ is a member of S with $(d-1)$-dimensional boundary $H^{=}$, let S^0 be the set of intersections of $H^{=}$ with the members of $S' = S - \{H^{\leq}\}$.

If $\cap(S^0) \neq \Phi$, then $\cap(S) \neq \Phi$. If $\cap(S^0) = \Phi$, then, by induction on the value of d, (2) says there is an $R' \subseteq S'$ of size at most d such that $[\cap(R')] \cap H^{=} = \Phi$.

By the Redundancy Lemma (3), either the set $R = R' \cup \{H^{\leq}\}$, of size at most $d+1$, is such that $\cap R = \Phi$, or else $H^{\leq}$ is redundant with R', and therefore also redundant with S'. In the latter case, by induction on the size of S, (2) says that either $\cap(S') \neq \Phi$, and hence $\cap(S) \neq \Phi$, or there is a subset $R \subseteq S' \subseteq S$, of size at most $d+1$, such that $\cap(R) = \Phi$.   □

PROOF OF THE REDUNDANCY LEMMA, (3). For every proper half-space $H^{\leq}$, there is a partition of E into three sets: the boundary $H^{=}$ of $H^{\leq}$, the open half-space $H^{<} = H^{\leq} - H^{=}$, and the open half-space $H^{>} = E - H^{\leq}$. Sets $H^{\geq} = H^{<} \cup H^{=}$ and $H^{\leq} = H^{<} \cup H^{=}$ are both proper half-spaces, intersecting in their common boundary $H^{=}$.

Every polyhedron, i.e., intersection of (closed) half-spaces, is topologically closed and connected.

Since $[\cap(R')] \cap H^{=} = \Phi$, polyhedron $\cap(R')$ is the disjoint union of polyhedron $R = [\cap(R')] \cap H^{\leq} \subseteq H^{<}$ and polyhedron $R_2 = [\cap(R')] \cap H^{\geq} \subseteq H^{>}$. If not (a), then $R \neq \Phi$, and if not (b), then $R_2 \neq \Phi$. However a connected set cannot be the disjoint union of non-empty closed sets, and so either (a) or (b).   □

The convexity of polyhedra can be nicely used in the proof of (3) instead of connectedness. Connectedness is used here in view of a later application of the same proof to topologically represented oriented matroids.

Let S be a finite set of half-spaces, $H_j^{\leq}$, of Euclidean space E. For every point p in E there is a vector of 'signs' with a component x_j for each member $H_j^{\leq}$ of S: $x_j = +$ if $p \in H_j^{<}$, $x_j = 0$ if $p \in H_j^{=}$, $x_j = -$ if $p \in H_j^{>}$, and one extra component, say x_0, always equals $+$. (x_0 corresponds to a half-space $H_0^{\leq}$ with its boundary 'at ∞' and with $H_0^{<} = E$. In 'affine space', i.e., Euclidean space, one stays on 'the plus side of ∞'. For clarification, see the next few paragraphs). The resulting finite set A of different sign-vectors is called a 'linearly representable affine matroid'.

The linearly representable affine matroid A nicely codifies the combinatorial structure of the way S determines a partitioning of E into 'relatively open polyhedra', each consisting of the points of E with the same sign-vector. The relatively open polyhedron $x \in A$ is 'a face' of the relatively open polyhedron $y \in A$, i.e., $x \preceq y$, when $x = y$ or when y can be changed into x by changing some non-zeros of y to 0. The union of the relatively open faces $\preceq y$ is the (closed) polyhedron $P(y)$ ('the relative closure of the relatively open polyhedron $y \in A$'), which is the intersection over all indices j, of the sets $H_j^{\leq}$, $H_j^{\geq}$, or $H_j^{=}$, according to whether $y_j = +$, $-$, or 0. The partial order $\preceq$ is isomorphic to the partial order $\subseteq$ of these closed polyhedra.

In the theory of linear inequality systems $Az \leq b$ it is often technically convenient to homogenize them to $[Az \leq bz_0, z_0 \geq 0]$. For each point (z, z_0), in say $d+1$ variables, we have a sign-vector x with a component corresponding to each of the inequalities; a component

of x is $+$, 0, or $-$, according to whether the corresponding inequality is strictly satisfied, satisfied as an equation, or not satisfied. In other words we have a set of half-spaces, say $H_j^{\leq}$, of a $(d+1)$-dimensional space, say E', such that each $H_j^{=}$ contains the origin. For each point $p \in E'$ there is a sign-vector x such that each component x_j is $+$, 0, or $-$, according to whether p is in $H_j^{<}$, $H_j^{=}$, or $H_j^{>}$. The resulting finite set of different sign-vectors, leaving out the vector of all zeros, is called a 'linearly representable oriented matroid'. Clearly the subset of these sign-vectors, for which $x_0 = +$, is a linearly representable affine matroid.

A *polytope P* is a bounded polyhedron. If in the proceeding paragraph we let P be any full-dimensional polytope such that the origin of the space of the points (z, z_0) is in the interior of P, then the various non-empty sets $H_j^{<}$, $H_j^{=}$, and $H_j^{>}$, and each of their various non-empty intersections, have non-empty intersections with the boundary, $bd(P)$, of P, because each linear ray from the origin intersects $bd(P)$ in exactly one point. The system of sets: $H_j^{<} \cap bd(P)$, $H_j^{=} \cap bd(P)$, and $H_j^{>} \cap bd(P)$, for all j; and the three sets of points in $bd(P)$ such that z_0 respectively is $>$, $=$, and $<$; is called a d-dimensional *linear pl-sphere system*. Its sign-vectors are the linearly representable oriented matroid of the preceding paragraph. The subsystem where $z_0 > 0$ is a d-dimensional *linear pl-affine system*; and the subsystem where $z_0 = 0$ is the corresponding '*pl-hypersphere at* ∞' (which is a $(d-1)$-dimensional linear pl-sphere system).

The set of sign-vectors of the linear pl-affine system is clearly the linearly representable affine matroid described earlier. Note that for any two polytopes P_1 and P_2 with (z, z_0) in their interior, and the two resulting linear pl-sphere systems, on P_1 and P_2 (as above on P), the linear rays from the origin determine, by their unique point intersections with $bd(P_1)$ and $bd(P_2)$, a 'piecewise linear' mapping between the two linear pl-sphere systems. Below we will define general pl-sphere systems, and general pl-affine systems, and make good use of them up to piecewise linear mappings. We will see that the Redundancy Lemma, Helly's Theorem, and their proofs, exactly generalize to pl-affine systems.

An *oriented matroid* is a finite set of sign-vectors which has many of the combinatorial properties of linearly representable oriented matroids. Several possible axiomatizations are presented in [2]. An *affine matroid* in general is then the subset of the sign-vectors of some oriented matroid M such that some coordinate, say x_o, is $+$. Not every oriented matroid, or affine matroid, is linearly representable. However the PL-Representation Theorem does state that every oriented matroid, and hence every affine matroid, is representable in the most natural *piecewise linear* way of generalizing a linear system, by a so-called 'pl-sphere system'; and the representation is unique up to piecewise linear mappings.

A finite union C of polytopes is called a *pl-ball* if it has a finite simplicial subdivision C' such that C has a bijection onto a polytope P which is linear on each simplex of C'. In other words C is a *pl-ball* if it has a finite simplicial subdivision C' which is isomorphic to a simplicial subdivision of P. A finite union C of polytopes is called a *pl-sphere* if it has a finite simplicial subdivision C' such that C has a bijection onto *the boundary* of a polytope P which is linear on each simplex of C'. In other words C is a *pl-sphere* if it has a finite simplicial subdivision C' which is isomorphic to a simplicial subdivision of the boundary of P. It is not true that if a finite union of polytopes is topologically a sphere (respectively, ball), then it is necessarily a pl-sphere (respectively, pl-ball). In pure 'piecewise linear topology' one never encounters either topological spheres as such, or geometric spheres.

'Combinatorial topology' means 'piecewise linear topology' because simple properties which one imagines about the topology of manifolds are valid using pl-mappings and not valid using general homeomorphisms. If you want your topology to look smooth that is o.k. as long as it is microscopically piecewise linear. Two of most important facts about pl-balls and pl-spheres are:

(4) A form of Newman's Theorem. *If the intersection of two d-dimensional pl-balls is a* $(d-1)$*-dimensional pl-ball on the boundary of each, then their union is a d-dimensional pl-ball. If the intersection of two d-dimensional pl-balls is the boundary of each, then their union is a d-dimensional pl-sphere.* (The first of these simple 'presumably obvious' facts is *not obvious* for pl-balls, and *not true* for topological balls even if they are finite unions of polytopes.)

A *hypersphere* $H^=$, in a d-dimensional pl-sphere K, is a $(d-1)$-dimensional pl-sphere $H^= \subset K$ such that $[K; H^=]$ is piecewise linearly equivalent (i.e., K and $H^=$ can both be piecewise linearly mapped by the same piecewise linear mapping) to $[bd(K'); bd(H')]$, where $bd(K')$ is the d-dimensional boundary of a $(d+1)$-dimensional polytope K' and $bd(H')$ is the $(d-1)$-dimensional boundary of the d-dimensional polytope H' which is the intersection of K' with a d-dimensional plane.

There is a d-dimensional pl-ball $H^\le \subset K$, and a d-dimensional pl-ball $H^\ge \subset K$, such $H^\le \cup H^\ge = K$ and $H^\le \cap H^\ge = H^=$. $H^\le$ and $H^\ge$ are called the proper (closed) half-space *sides* of $H^=$ in K. (One cannot say that every $(d-1)$-dimensional pl-sphere contained in a d-dimensional pl-sphere K is a hypersphere in K. However the usual form of 'Newman's Theorem' states that if a d-dimensional pl-ball B_1 is contained in a d-dimensional pl-sphere K, then its boundary $bd(B_1)$ is a hypersphere in K, and $(K-B_1) \cup bd(B_1) = B_2$ is a d-dimensional pl-ball with boundary $bd(B_2) = bd(B_1)$.)

A *d-dimensional pl-sphere system*, Q, is a d-dimensional pl-sphere K; a finite index set J; and 'the half-spaces and hyperspheres of Q': for each $j \in J$, either an improper half-space $H_j = K$; or a proper half-space $H_j^\le$ of K, and its companion $H_j^\ge$, and the hypersphere $H_j^= = H_j^\le \cap H_j^\ge$ of K; such that:

(K1) The intersection of each subset of the half-spaces of Q is a pl-ball or pl-sphere.
(K2) The intersection of each subset of the hyperspheres, $H_j^=$, of Q is a pl-sphere (of dimension from -1 to $d-1$), called a *flat* of Q.
(K3) For any flat F of Q and any half-space say $H_j^\le$ of Q, if $F \subset H_j^=$, then $F \cap H_j^\le = F$ is an improper half-space of F, indexed by $j \in J$. If F is not contained in $H_j^=$, then the flat, $F \cap H_j^=$, is a hypersphere in F, indexed by $j \in J$, having proper half-space sides $F \cap H_j^\le$ and $F \cap H_j^\ge$. (Note that any flat F of Q together with its half-spaces indexed by $j \in J$ is a pl-sphere system.)

From any pl-sphere system Q, by choosing a particular $H_0^=$ (where $0 \in J$) to be our 'hypersphere at infinity', and choosing $H_0^<$ to be our affine space E, then for any $S^+ \subseteq J - \{0\}$, and $S^- \subseteq J - \{0\}$, we can apply the proof of (3), the Redundancy Lemma, and the proof of (2), Helly's Theorem for half-spaces, directly to the set $S = [H_j^\le \cap H_0^< : j \in S^+] \cup [H_j^\ge \cap H_0^< : j \in S^-]$ of 'piecewise linear half-spaces' of E. An $[E, S]$ obtained in this way is called a *pl-affine system*. In order not to need to change the wording in the proof of (2), we also want to allow the S of a pl-affine system to include copies of E, or of Φ, as improper half-spaces because they can arise in the inductive process of intersecting a proper half-space with the hyperplane boundary of another. (In the case of Φ arising, there is a non-empty intersection at $H_0^=$, the ∞ of the affine E.)

(5) *Thus* (3), *the Redundancy Lemma, and* (2), *Helly's Theorem For Half-Spaces, are proved for any pl-affine system.*

For any d-dimensional sphere system, Q, each point p in K has a sign-vector $x = (x_j : j \in J)$ such that $x_j = 0$ if $H_j = K$ or if $p \in H_j^=$; $x_j = +$ if $p \in H_j^<$; $x_j = -$ if $p \in H_j^>$. Note that the sign-vectors of the pl-sphere system of any flat of Q is simply the

subset of the sign-vectors of Q for which certain components are 0. The -1-dimensional flat is Φ; it has no sign-vectors. Note that we have no need for Φ as a possible improper half-space of a non-empty pl-sphere system. From axioms for oriented matroids, it is easy to show that the set $M(Q)$ of sign-vectors thus determined by Q is an oriented matroid. Conversely:

(6) The PL-Representation Theorem For Oriented Matroids (Lawrence, Edmonds and Mandel, 1978). *Every (rank $d+1$)-oriented matroid M is the $M(Q)$ of a (d-dimensional) pl-sphere system Q such that $\cap(H_j^= : j \in J)$, the intersection of all the hyperspheres of Q, is empty; Q is unique up to piecewise linear mappings.*

The books [1] and [2] contain proofs of (6) which are essentially the same as the proof by Edmonds and Mandel. It uses Newman's Theorem (4) to paste together pl-balls according to oriented-matroidal structure. The variant of (6) by Jim Lawrence using topological spheres is simpler, though it says less because fundamental combinatorial properties such as Newman's Theorem are not valid.

For an affine matroid A, determined by the oriented matroid M of pl-sphere system Q with $0 \in J$ as the specified infinity coordinate, A is the subset of the sign-vectors of M such that $x_0 = +$. Where $h \in J - \{0\}$, the *half-space h^+ (or h^-)* of affine matroid A is the subset of the sign-vectors of M such that $x_0 = +$, and $x_h = +$ or 0 (respectively, $x_h = -$ or 0). Just as a polyhedron in Euclidean space E is the intersection of a finite set of its half-spaces, a *polyhedron* of the affine matroid A is the intersection of a subset of its half-spaces. Where $h \in J$, the *half-space h^+ (or h^-)* of oriented matroid M is the subset of sign-vectors of M such that $x_h = +$ or 0 (respectively, $x_h = -$ or 0). Simple examples show that statement (7) with 'affine matroid A' replaced by 'oriented matroid M' is not true.

(7) A Helly's Theorem For Oriented Matroids. *For any subset S of the half-spaces (or of the polyhedra) of an, at most rank $d + 1$, affine matroid A, either $\cap(S) \neq \Phi$, or there is a subset $R \subseteq S$ of size at most $d + 1$ such that $\cap(R) = \Phi$.*

Theorem (7) follows immediately from (5) and (6). It can be proved directly and easily without (5) and (6) by an abstract oriented-matroidal proof which is analogous to the proof of (2) and which is much easier than proving (6). However, Theorem (6) has the advantage of providing a geometric setting for all of oriented matroid theory which is exactly true, rather than only analogous to geometric statements.

Abstract oriented matroid theory, such as (7), can be formally a convenient setting for proving statements of combinatorial topology, such as (5), in the way that Cartesian coordinates can be formally a convenient setting for proving statements of classical geometry. However we will here stick to explanations in terms of the combinatorial topology in order to visually elucidate the formalities. We present now what is perhaps a novel strengthened form of (2), and its relationship to Farkas' Lemma. Farkas' Lemma is well known as a more practical characterization than (2′) of when a finite set, $Az \leq b$, of linear inequalities does not have a solution, z. It states that:

(8) *A finite set $Az \leq b$ of linear inequalities has a solution z or else there is a vector $y \geq 0$ such that $yA = 0$ and $yb < 0$.* (Not both, since then $(yA)z \leq yb$ would need to be satisfied by both the z and y together.)

Several known ways of proving (8) indeed also prove the following strengthened form, (9), which immediately implies Helly (2′) (and thus provides an alternative proof of Helly's Theorem for general convex sets).

(9) *A finite set $Az \leq b$ of linear inequalities in the d variables $z = (z_1, \ldots, z_d)$ has a solution z or else there is a subset $A'z \leq b'$ of at most $d + 1$ of the inequalities and a vector $y \geq 0$ such that $yA' = 0$ and $yb' < 0$.*

Helly $(2')$ is more elegant in some ways than (8) or (9). We find that $(2')$ 'almost implies' (9), but not quite. Theorem (10) has the following properties.

(a) does not involve any numbers like those of the y in (9);
(b) is obviously a strengthening of Helly $(2')$;
(c) implies, by simple algebra, a strengthening of (9);
(d) extends to a theorem about pl-sphere systems, which
(e) can be proved by a slight strengthening of the topological way we proved (2) and (5).

(10) Theorem. *A finite set $Az \leq b$ of linear inequalities has a solution z or else there is a subset $A'z \leq b'$ which has no solution z and is such that the number of inequalities in $A'z \leq b'$ is exactly one greater than the rank of A'.*

Since (rank of A') $\leq$ (number of columns of A), we have (b).

Let us see (c). Each inequality, say $a_j z \leq b_j$, of the system $A'z \leq b'$ in (10), is equivalent to $a_j z + x_j = b_j$, $x_j \geq 0$. From the system $A'z + Ix = b'$ of these equations, derive by row operations an equivalent system, L, where one of the equations $yA'z + yx = yb'$ is such that $yA' = 0$; and where each other equation of system L, say equation i, involves a component, say z_i, of z with a non-zero coefficient, where z_i does not appear with non-zero coefficient in any other equation of L. There exists such an L because the number of rows of A' is exactly one greater than the rank of A'. It is clear from system L that if the single equation $yx = yb'$ can be satisfied by some $x \geq 0$, then $A'z \leq b'$ is satisfied by a certain z. Clearly the only way that $yx = yb'$ can be not satisfiable by any $x \geq 0$ is for yb' to be non-zero and for each component of y to not have the same non-zero sign as yb'. That is we can take y to be such that $yb' < 0$ and $y \geq 0$. Thus from (10) we have the following strengthening of (9).

(11) *A finite set $Az \leq b$ of linear inequalities has a solution z or else there is a subset $A'z \leq b'$, such that the number of inequalities in $A'z \leq b'$ is one greater than the rank of A', and a vector $y \geq 0$ such that $yA' = 0$ and $yb' < 0$.*

With regard to (d), note that the rank of A' is one less than the rank of the matrix consisting of $[A' \mid -b']$ with one additional row which is all zeros except for -1 in the same column as b'. In other words the rank of A' is one less than the rank of the matrix of coefficients of the homogeneous system $\{A'z - b'z_0 \leq 0, -z_0 \leq 0\}$. Therefore in a pl-sphere system Q corresponding to $\{Az - bz_0 \leq 0, -z_0 \leq 0\}$, where $H_0^<$, corresponding to $-z_0 < 0$, is our affine space E, we interpret the rank of A' as one less than 'the rank' of the subset J' of the hyperplanes which consists of hyperplane H_0 and the hyperplanes corresponding to the equations of $A'z - b'z_0 = 0$.

(12) *The rank of a subset J' of the hyperspheres of a pl-sphere system Q (or the rank of a set of half-spaces of Q whose set of boundaries is J') is the dimension of Q minus the dimension of the flat $\cap(J')$. In particular, when the intersection of all the hyperspheres of Q is Φ, having dimension -1, the rank of the set of all the hyperspheres of Q, i.e., the rank of the oriented matroid represented by Q, is the dimension of Q plus one.*

We will not prove here the claim that the rank of J' as described in (12) is in fact the rank of the corresponding subset of elements of the oriented matroid which Q represents, and that, in the case that Q is the pl-sphere system corresponding to a system of homogeneous linear inequalities, the rank of J' as described in (12) is the rank of the matrix of coefficients of the corresponding subset of homogeneous inequalities. Assuming this claim, Theorem (10) is a special case of the following theorem.

(13) Theorem. *For any pl-affine system, consisting of pl-sphere system Q and the open half-space $H_0^<$ of Q as the affine space, and for any subset S of the half-spaces of Q, either*

$\cap(S \cup \{H_0^<\}) \neq \Phi$ *or there is some* $R \subseteq S$ *such that* $\cap(R \cup \{H_0^<\}) = \Phi$, *and such that the rank of* $R \cup \{H_0^\leq\}$ *equals the cardinality of* R.

PROOF (e). The inductive proof of (13) is anchored either where $\cap(S \cup \{H_0^<\}) \neq \Phi$, or where some member of S is a half-space, say $H_1^\leq$, of Q such that $H_1^\leq \cap H_0^< = \Phi$. In this latter case, let $R = \{H_1^\leq\}$. The dimension of $H_1^= \cap H_0^= = H_0^=$ is one less than the dimension of Q, and so the rank of $R \cup \{H_0^\leq\}$ and the cardinality of R are both one.

Otherwise there is some $H^\leq \in S$ such that $H^\leq \cap H_0^<$ is a proper half-space of affine space $H_0^<$ which has a non-empty boundary $H^= \cap H_0^<$ in $H_0^<$. Assume (13) is true for the affine pl-system obtained by restricting Q and $H_0^<$ to $H^=$. The proof continues as the proof for (2) and (5), except that we observe that when we obtain some R' such that $\cap(R' \cup \{H_0^<\}) \cap H^\leq = \Phi$, and such that the rank of $R' \cup \{H_0^\leq\}$ restricted to $H^=$ equals the cardinality of R', then $R = R' \cup \{H^\leq\}$ is an R as described in (13) because, since the dimension of the pl-sphere system goes up by one, and since the intersection of $H_0^=$ and the boundaries of members of R is the same as the intersection of $H_0^=$ and the boundaries of the members of R', the rank of $R \cup \{H_0^\leq\}$ in Q equals the cardinality of R. $\qquad\square$

ACKNOWLEDGEMENT

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