Scarf Oiks

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Abstract

We formulate the famous Scarf Lemma in terms of oiks. This lemma has two fundamental applications in game and graph theory. In 1967, Scarf derived from it core-solvability of balanced cooperative games. Recently, it was shown that kernel-solvability of perfect graphs also results from this lemma.

We show that Scarf's combinatorially defined oiks are in fact realized by polytopes, and that Scarf's algorithm for proving the Scarf Lemma is an instance of the Lemke-Howson algorithm for finding an equilibrium of a bimatrix game.

We give a sequence of inputs of two equal d-dimensional Scarf oiks on $2d$ vertices, found by computer, such that the pivoting path of the algorithm grows exponentially with d.

Keywords: Euler complex (oik), room, wall, exchange algorithm, pivot; Scarf Lemma, balanced games; kernel, kernel-solvability, perfect graph.

1 Introduction

The concept of room-partitioning of an oik (Euler complex) was introduced in [2] (see also [4] and [5]).

Given two integers n and d such that $n > d > 1$, a *d-dimensional complex* $\mathcal{O} = (V, \mathcal{R})$ is a uniform hypergraph of edge-size d on the ground set V of cardinality n. Standardly, the elements $v \in V$ are called *vertices*, while the edges $R \in \mathcal{R}$ are called *rooms*; each room consists of d vertices.

Furthermore, given a room R and a vertex $v \in R$, the difference $W = R \setminus \{v\}$ (of cardinality $d-1$) is called a *wall*. A complex is called an *oik* if each wall W is contained in a *positive even* number $k(W)$ of rooms.

Two rooms $R, R' \in \mathcal{R}$ are called *adjacent* if their intersection is a wall, or in other words, if their symmetric difference $R \Delta R'$ is a pair of vertices $v \in R$ and $v' \in R'$.

An oik $\mathcal{O}$ will be called 2-*adjacent* if $k(W) \equiv 2$ for every wall W, that is, if each wall is contained in exactly two (adjacent) rooms.

In this paper, we show that a famous construction by Herbert Scarf [10] (see also [8,9,6]) is in fact a 2-adjacent oik.

An $m \times (m+n)$ real non-negative matrix A is called a *Scarf* matrix if

(i) $m \geq 2$ and $n \geq 1$;

(ii) $a(i,j) > a(i, m+k) > a(i,i) \geq 0$ for all $i, j \in [m] = \{1, \ldots, m\}$, where $i \neq j$, and $k \in [n] = \{1, \ldots, n\}$;

(iii) $a(i, m+k) \neq a(i, m+\ell)$ for all $i \in [m]$ and distinct $k, \ell \in [n]$.

Let us consider the following example in which $m = 3$ and $n = 4$:

$$\begin{matrix} 0 & M & M & 1 & 2 & 3 & 4 \\ M & 0 & M & 3 & 1 & 2 & 4 \\ M & M & 0 & 4 & 3 & 2 & 1 \end{matrix}$$

A $m \times (m+n)$ Scarf matrix will be called *canonical* if

(iv) $a(i,i) = 0$ for any $i \in [m]$ and $a(i,j) = M > n$ for any distinct $i, j \in [m]$;

(v) in each row, the last n entries form a permutation of $[n] = \{1, \ldots, n\}$.

Given a constant $M > n$, there are $(n!)^m$ canonical Scarf matrices.

Let $V = [m+n]$ be the set of columns of a Scarf (not necessarily canonical) matrix A. A subset $J \subseteq V$ is called *dominating* if for each column $k \in [m+n]$ there is a row $i \in [m]$ such that $a(i,k) \leq a(i,j)$ for each $j \in J$.

The next four properties (vi - ix) obviously hold whenever J is dominating:

(vi) If $J' \subseteq J$ then J' is dominating, too.

In other words, domination is a hereditary property defined on V.

Each column $j \in J$ is dominated by J, as well as any other column. When $J \not\subseteq [m]$, let j_i denote the column that realizes the minimum of the entries of row $i \in [m]$ among the columns in J. This results in the following claim.

(vii) For each column $j \in J$, there exists at least one row i such that $j_i = j$.

Indeed, otherwise the column j is not dominated by J.

Obviously, a minimum of $a(i,j), j \in J$ can appear more than once only when $J \subseteq [m]$. This observation and (vii) imply the following two claims:

(viii) $|J| \leq m$.

(ix) If $|J| = m$ then each row $i \in [m]$ of the $[m] \times J$ submatrix A_J. has a unique minimum $a(i, j_i)$ and these m minima form a permutation in A_J.

The following simple observation will play an important role.

Lemma 1 *In a Scarf matrix A, the first m columns, $J = [m]$, do not form a dominating set, while each proper subset $J \subset [m]$ is a dominating set.*

It will be convenient to call $J = [m]$ a *special dominating* set. It is easy to verify that the four properties (vi - ix) still hold after this extension. Moreover, due to it, the following key statement becomes true.

Theorem 1 *Each dominating set of $m-1$ columns is contained in exactly two dominating sets of m columns; one of these two sets might be $[m]$.*

Each dominating set of $m-1$ (respectively, m) columns, including the special one) constitutes a wall (respectively, room). By Theorem 1, every wall is contained in exactly two rooms, so this structure defines a 2-adjacent oik of dimension $d = m$. By the general room partitioning theorem for Oiks [2], it follows that given a room partitioning, one can construct another one by traversing the exchange graph, by a sequence of pivots, see, for example [3] or [5] for the definitions and more details. In the special case of "Scarf oiks", this result was obtained in [8,9].

The following example shows that properties (i,ii,iii), that define the Scarf $m \times (m+n)$ matrices, can hardly be relaxed; even a slight modification of them might destroy the oik-structure.

Example 1 *Let us consider the following "almost" Scarf matrix.*

$$\begin{matrix} 0 & 1 & 2 \\ 2 & 0 & 1 \end{matrix}$$

It is easy to verify that columns 1 and 3 form a dominating set, while 2 and 3 do not, since for them both row-minima are in the column 2. Whether $\{1, 2\}$ is a special dominating set or it is not, still the oik properties fail. Indeed, set $\{3\} = \{1, 3\} \setminus \{1\}$ should be a wall, yet, it is contained in only one room.

The first proof of Theorem 1 given in [10] was then simplified in [6] and later in [1].

2 Scarf's oiks are polytopal

Example 2 : Polytopal oiks. *Let $Ax = b, x \geq 0$ be a tableau, as in the simplex method, that is, A is a $m \times n$ matrix that contains an $m \times m$ identity submatrix and all coordinates of $b \in \mathbb{R}^m$ are strictly positive. Let us also* *assume that the solution set is bounded and all basic feasible solutions are non-degenerate.*

Let V be the column set of A. By definition, its subset $R \subseteq V$ is a room if and only if $V \setminus R$ is a basis of the tableau. The hypergraph $\mathcal{O} = (V, \mathcal{R})$ of the rooms defines an oik of dimension $d = n - m$. This results from the following exchange property of the bases. Given a basic set of columns in A (the complement to a room), let us add to it an arbitrary "entering" column (thus getting the complement to a wall). Then there exists a unique "leaving" column such that all coefficients of the right-hand-side remain positive.

Combinatorially the above oik is defined by the boundary of an $(n-m)$-dimensional simplicial polytope.

Given a $m \times (m+n)$ Scarf matrix A (not necessarily in canonical form) defined by formulae (i,ii,iii), introduce a $n \times (m+n)$ matrix B as follows:

(i) $b(i, m+j) = \delta_i^j$ for $i, j \in [n]$,

where standardly $\delta_i^j = 1$ if $i = j$ and $\delta_i^j = 0$ if $i \neq j$; in other words, the last n columns of B form the $n \times n$ identity matrix;

(ii) $b(j, i) = 1 - a(i, m+j)^{-K}$; $i \in [m]$, $j \in [n]$;

In other words, the last n columns of A form a $m \times n$ matrix C and the first m columns of B form a $n \times m$ matrix D such that

(iii) $d(i, j) = 1 - c(j, i)^{-K}$; $i \in [n]$, $j \in [m]$, where $K > 0$ is a large constant.

The transformation $f(a) = 1 - a^{-K}$ was suggested by Scarf in [8]. Obviously,

(iv) f is a monotone increasing function for any fixed positive K;

(v) $f(a) = 1 - 1/a^K > 0$ when $a > 1$ and $f(a) \leq 0$ when $0 < a \leq 1$.

Due to (v), it will be convenient to assume that

(vi) $a(i, m+j) > 1$ for all $i \in [m], j \in [n]$, in the considered Scarf matrix A.

It is clear that (vi) can be assumed without any loss of generality, since the Scarf oik $\mathcal{O}_A$ remains the same if we add a constant to these entries of A.

Furthermore, by construction,

(vii) the matrices A and B have the common column-set $V = [m+n]$.

Let us consider the system of n equations $Bx = e_n$ of $m+n$ real variables $x \in \mathbb{R}^{m+n}$, where $e_n \in \mathbb{R}^n$ is the vector of n ones. Standardly, a set of columns $J \subseteq V$ is called *basic* if $Bx = e_n$ for a non-negative x such that $x_j = 0$ whenever $j \notin J$. [Conversely, $x_j > 0$ for $j \in J$, because matrix B is not degenerate for any Scarf matrix A and sufficiently large K.] Obviously, set $J_0 = \{m+j \mid j \in [n]\}$ of the last n columns in B is basic. Moreover, it is well known that each basic set J can be obtained from J_0 by a sequence of exchanges produced by simplex-method; see Example 2. In that example, we assigned the room $V \setminus J$ to each basic set J in B thus getting an m-dimensional oik $\mathcal{O}_B$. The next theorem shows that A and B generate the same oik, $\mathcal{O}_A = \mathcal{O}_B$. In other words, that the Scarf oik $\mathcal{O}_A$ is polytopal.

Theorem 2 *A column-set J is basic in B if and only if the complementary set $V \setminus J$ is dominating in A, provided $K > 0$ is sufficiently large.*

The proof can be found in [4].

3 Exchange paths of exponential lengths

The exchange path between two room-partitions may be exponential:

(i), [5], in the number of vertices and in the number of explicitly given rooms, already for dimension $d = 3$

or

(ii), here, in dimension d already for $2d$ vertices, or in other words, for only 2 rooms in a partition, however with an exponential number of implicit rooms.

In the construction for (i), given in [5], for each $k = 3, 4 \ldots$, there is a 3-dimensional oik $\mathcal{O}_k$ defined by $12(k-2)$ rooms (triangles) on $n = 3k$ vertices. (Hence, each room-partition consists of k rooms.) This oik has an exchange path of length $7 \times 2^{k-1} - 5$ between two room-partitions.

Here, for each $d = 2, 3, \ldots$ we shall construct a d-dimensional Scarf oik $\mathcal{O}_d$ with $n = 2d$ vertices. (Hence, each room-partition consists of only two rooms.) This oik has an exchange path of length $3 \times 2^{d-2} - 1$ between two room-partitions. The oiks $\mathcal{O}_d$ for $d = 2, 3, \ldots$ are given by the following $d \times 2d$ Scarf's tables T_d, where, as usual, M is a very large number:

$$\begin{array}{ccc} 0 & M & 1\,2 \\ M & 0 & 2\,1 \end{array} \quad \text{for } d = 2; \qquad \begin{array}{cccc} 0 & M & M & 1\,2\,3 \\ M & 0 & M & 3\,1\,2 \\ M & M & 0 & 3\,2\,1 \end{array} \quad \text{for } d = 3;$$

$$\begin{array}{cccccccccc} 0 & M & M & \dots M & 1 & 2 & 3 & \dots d-1 & d \\ M & 0 & M & \dots M & d & 1 & 2 & \dots d-2 & d-1 \\ M & M & 0 & \dots M & d & d-1 & 1 & \dots d-3 & d-2 \\ \vdots & & & & & & & & \\ M & M & \dots & 0 \; M & d & d-1 & d-2 & \dots \; 1 & 2 \\ M & M & \dots & M \; 0 & d & d-1 & d-2 & \dots \; 2 & 1 \end{array} \qquad \text{for an arbitrary } d \geq 2.$$

As before, $V_d = \{1, \dots, d, d+1, \dots, 2d\}$ is the set of columns of table T_d. We begin with the following partition of V_d into two rooms:

$$V_d = R_1^0 \cup R_2^0 = \{1, \dots, d\} \cup \{d+1, \dots, 2d\}$$

and eliminate column 1 from room R_1^0 getting the wall $W_1^0 = R_1^0 \setminus \{1\} = \{2, \dots .d\}$. It is not difficult to verify that the entering column is $2d$ and, hence, the adjacent room is $R_1^1 = \{2, \dots, d, 2d\}$.

Since rooms R_1^1 and R_2^0 form a butterfly (see [3]) with the intersection $2d$, next, we eliminate $2d$ from R_2^0 getting the wall $W_2^0 = \{d+1, \dots, 2d-1\}$. The entering column is $d-1$ and, hence, the adjacent room is $R_2^1 = \{d-1, d+1 \dots, 2d-1\}$.

Since rooms R_1^1 and R_2^1 form a butterfly with the intersection $d-1$, next, we eliminate $d-1$ from R_1^1 getting the wall $W_1^1 = \{2, \dots, d-1, 2d\}$. Then, the entering column is $2d-2$, etc., until we obtain another room partition.

Since each room-partition consists of only two rooms, the exchange path is uniquely defined by the sequence S_d of the leaving (or entering) columns. (Obviously, they leave (and enter) the first and second rooms alternatingly.) The tables S_d can be conveniently represented as follows:

1
6 2 4
6 for $d = 3$

resulting after 5 steps in the partition $\{1, 3, 4\} \cup \{2, 5, 6\}$;

1
8 3 6
8 5 2
8 6 3
8 for $d = 4$

resulting after 11 steps in the partition $\{1, 3, 4, 5\} \cup \{2, 6, 7, 8\}$;

1			
10	4 8		
10		7 3	
10	8 4		
10			6 2
10	4 8		
10		3 7	
10	8 4		

10 for $d = 5$

resulting after 23 steps in the partition $\{1, 3, 4, 5, 6\} \cup \{2, 7, 8, 9, 10\}$; ...

1				
$2d$	$d-1$ $2d-2$			
$2d$		$2d-3$ $d-2$		
$2d$	$2d-2$ $d-1$			
$2d$			$2d-4$ $d-3$	
$2d$	$d-1$ $2d-2$			
$2d$		$d-2$ $2d-3$		
$2d$	$2d-2$ $d-1$			
...				
$2d$				$d+1$ 2
...				
$2d$	$d-1$ $2d-2$			
$2d$		$2d-3$ $d-2$		
$2d$	$2d-2$ $d-1$			
$2d$			$2d-4$ $d-3$	
$2d$	$d-1$ $2d-2$			
$2d$		$d-2$ $2d-3$		
$2d$	$2d-2$ $d-1$			

$2d$ for an arbitrary $d \geq 2$, resulting after $3 \times 2^{d-2} - 1$ steps in the partition $\{1, 3, 4, \ldots d+1\} \cup \{2, d+2, \ldots 2d\}$ which can be obtained from the original partition by the exchange of 2 and $d+1$.

This example arose from a computer search. We examined (up to symmetry) all the canonical Scarf matrices of size 3×6 and 4×8, and looked for the matrices with the longest exchange path, leading to the above construction.

Remark 3.1 *Let us also remark that no exponential in d example can exist for a pair of d-dimensional Sperner oiks with $2d$ vertices in each. Morris [7] proved that in this case any exchange path between two room-partitions is of length at most $2d$. Moreover, it is not difficult to verify that there are exactly $(d-1)!$ paths of length $2d$. They connect the following room-partitions: Given $2d$ vertices, let us color them $\{1, 2, \ldots, d; 1, 2, \ldots, d\}$ in the first Sperner* *oik and* $\{1, 2, \ldots, d; \sigma(1), \sigma(2), \ldots, \sigma(d)\}$ *in the second one, where* σ *is a* d*-permutation. Let us consider two rooms induced by the first and second* d *vertices in the first and second oiks, respectively. It is easily seen that an exchange path beginning in this room-partition is of length* $2d$ *whenever permutation* σ *is prime (i.e., formed by a single cycle) and of length* $< 2d$ *otherwise.*

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