Sperner Oiks

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Abstract

The idea of "Lemke pivoting in a family of oiks (Euler complexes)" generalizes, and abstracts to pure combinatorics, the Lemke-Howson exchange algorithm for finding a Nash equilibrium in bimatrix games, as well as the classical algorithm for finding the properly colored room in Sperner's Lemma. Given a "room-partitioning", this algorithm finds another (distinct) room-partitioning by traversing the exchange graph. In this paper we show that each family of k oiks $\mathcal{O} = \{\mathcal{O}_1, \ldots, \mathcal{O}_k\}$ can be reduced to a pair of oiks $\mathcal{O}' = \{\mathcal{O}_1 + \ldots + \mathcal{O}_k, \mathcal{O}_0\}$ (one of which, $\mathcal{O}_0$, is a Sperner oik) such that the exchange graphs for $\mathcal{O}$ and $\mathcal{O}'$ are isomorphic. Numerous application of Sperner's Lemma in combinatorial topology are well known.

Keywords: Euler complex (oik), room, wall, manifold, exchange algorithm, pivot; Lemke-Howson algorithm; Sperner Lemma, Brouwer Theorem, KKM-Theorem

1 Introduction

1.1 Oiks; definition and examples

The concept of an *oik* (short for Euler complex) was recently introduced in [1] as follows. Given two integers n and d such that $n > d > 1$, a *d-dimensional complex* $\mathcal{O} = (V, \mathcal{R})$ is a uniform hypergraph of edge-size d on the ground set V of cardinality n. Standardly, the elements $v \in V$ are called *vertices*, while the edges $R \in \mathcal{R}$ are called *rooms*; each room consists of d vertices.

Furthermore, given a room R and a vertex $v \in R$, the difference $W = R \setminus \{v\}$ (of cardinality $d-1$) is called a *wall*. A complex is called an *oik* if each wall W is contained in a *positive even* number $k(W)$ of rooms.

Two rooms $R, R' \in \mathcal{R}$ are called *adjacent* if their intersection is a wall, or in other words, if their symmetric difference $R \Delta R'$ is a pair of vertices $v \in R$ and $v' \in R'$.

An oik $\mathcal{O}$ will be called 2-*adjacent* if $k(W) \equiv 2$ for every wall W, that is, if each wall is contained in exactly two (adjacent) rooms.

The next four examples of 2-adjacent oiks are borrowed from [1].

Example 1 : Pseudo-manifolds. *A $(d-1)$-dimensional simplicial pseudo-manifold is a d-dimensional oik in which each d vertices are contained in exactly zero or two rooms; in other words, each wall is in exactly two rooms.*

An important special case is a triangulation of a compact manifold M, oriented or not. In particular, if M is a $(d-1)$-dimensional sphere, the corresponding oik $\mathcal{O}(M)$ is realized by a d-dimensional polytope whose every facet is a simplex with d vertices.

The oiks generated by pseudo-manifolds, manifolds, and polytopes will be called PM-, M-, and P-oiks, respectively. The latter will be also called *polytopal* and represented as follows.

Example 2 : Polytopal oiks. *Let $Ax = b, x \geq 0$ be a tableau, as in the simplex method, that is, A is a $m \times n$ matrix that contains an $m \times m$ identity submatrix and all coordinates of $b \in \mathbb{R}^m$ are strictly positive. Let us also assume that the solution set is bounded and all basic feasible solutions are non-degenerate.*

Let V be the column set of A. By definition, subset $R \subseteq V$ is a room if and only if $V \setminus R$ is a basis of the tableau. The hypergraph $\mathcal{O} = (V, \mathcal{R})$ of the rooms defines an oik of dimension $d = n - m$. This results from the following exchange property of the bases. Given a basic set of columns in A (the complement to a room), let us add to it an arbitrary "entering" column (thus getting the complement to a wall). Then there exists a unique "leaving" column such that all coefficients of the right-hand-side remain positive.

Combinatorially the above oik is defined by the boundary of an $(n-m)$-dimensional simplicial polytope.

Remark 1 *The boundary (surface) of a simplicial polytope of dimension d is a manifold of dimension $d-1$. Thus, the corresponding oik can be called either d- or $(d-1)$-dimensional. Respectively, there are two options: to call an oik d-dimensional when its rooms are of cardinality d or $d+1$. Here we chose the first option, while the second one is chosen in [1].*

Let us consider two examples of special polytopal oiks.

Example 3 : Gale oiks. *Let us consider Gale's cyclic polytope* $P = P(d, n) \subseteq \mathbb{R}^d$ *with* n *vertices. In [4], David Gale proved that the rooms of the corresponding oiks are defined by the* **cyclic** *binary* n*-vectors* $x \in \{0, 1\}^n$ *with* d *ones such that the following* **Gale evenness condition** *holds: If* d *is even then all sequences of successive ones in* x *are even. (Let us remark that the first and the last such sequences in* x *make one sequence* s_0*, since* x *is cyclic.) If* d *is odd then all above sequences are still even, except* s_0*, which must be odd.*

Example 4 : Sperner oiks. *Let the* n *elements of a set* V *be colored by* d *colors, where* $d < n$*. A subset* $R \subset V$ *is a room if and only if* $V \setminus R$ *contains exactly one vertex of each color.*

The defined hypergraph $\mathcal{O} = (V, \mathcal{R})$ *is an oik of dimension* $n - d$*.*

Indeed, the complement to a wall, which is colored $\{1, 2, \ldots, d, j\}$*, contains exactly two complements to rooms, which are colored* $\{1, 2, \ldots, d\}$*.*

This oik is polytopal. In particular, when V *consists of* $2d$ *vertices and each color appears twice,* $\{1, 1, 2, 2, \ldots, d, d\}$*, the corresponding polytope is polar to the* d*-dimensional cube. We leave the proofs to the reader.*

Remark 2 *In the latter case the complement to a room is also a room. However, in general, such a claim does not hold for the above four examples.*

In [3], we introduce one more family of 2-adjacent oiks based on the Scarf Lemma [11] and we prove that these oiks are polytopal.

1.2 Finding another room-partition or room-selection of fixed degrees by the exchange algorithm

An *oik-family* is a set of k oiks $\mathcal{O} = \{\mathcal{O}_1, \ldots, \mathcal{O}_k\}$ (of dimensions $d_1, \ldots, d_k$) defined on the same vertex-set V. Some of these oiks may be isomorphic.

Given an oik-family $\mathcal{O}$, a room-selection is a hypergraph $\mathcal{R} = \{R_1, \ldots, R_k\}$ in which R_i is a room of oik $\mathcal{O}_i$ for all $i \in [k] = \{1, \ldots, k\}$. Standardly, $\deg_{\mathcal{R}}(v)$ denote the degree of a vertex $v \in V$ in $\mathcal{R}$, that is, the number of rooms of $\mathcal{R}$ that contain v. A room-selection $\mathcal{R}$ is called *a room-partition* if $\deg_{\mathcal{R}}(v) \equiv 1$ for each $v \in V$. It was shown in [1] that every oik-family has an even number of room-partitions.

Remark 3 *Let us note, however, that this number may be* 0*. Moreover, it might be NP-hard to verify the existence of a room-partition.*

Furthermore, given a room-partition, an **exchange algorithm** to get another one is proved in [1]. This algorithm is based on constructing and, then, traversing the *exchange graph*. Given a family of oiks $\mathcal{O} = \{\mathcal{O}_1, \ldots, \mathcal{O}_k\}$ on the common vertex-set ($\mathcal{O}_i = (V, \mathcal{R}_i)$, $i \in [k] = \{1, \ldots, k\}$), let us fix a special vertex $w \in V$ and define the exchange graph $\mathcal{G} = \mathcal{G}(\mathcal{O}, w)$ as follows.

A room-selection $\mathcal{R}$ is called a *skew room-partition* or a *butterfly* if $\deg_{\mathcal{R}}(w) = 0$, $\deg_{\mathcal{R}}(u) = 2$ for a unique vertex $u \in V$, and $\deg_{\mathcal{R}}(v) \equiv 1$ for all other vertices $v \in V \setminus \{u, w\}$. Let $\mathcal{V}$ and $\mathcal{V}_1$ denote the sets of all room-partitions and skew room-partitions, respectively.

Two room-selections $\mathcal{R} = \{R_1, \dots, R_k\}$ and $\mathcal{R}' = \{R'_1, \dots, R'_k\}$ are called *adjacent* if their symmetric difference $\mathcal{R}\Delta\mathcal{R}'$ is a pair of **adjacent** rooms (R_i, R'_i) from $\mathcal{O}_i$ for some $i \in [k]$. If also $\mathcal{R}, \mathcal{R}' \in \mathcal{V} \cup \mathcal{V}_1$ then $(\mathcal{R}, \mathcal{R}') \in \mathcal{E}$. Thus, the exchange graph $\mathcal{G}(\mathcal{V} \cup \mathcal{V}_1, \mathcal{E})$ is defined.

It is easy to list all rooms adjacent to a given room R of a given oik $\mathcal{O}$. To do so, let us select a vertex $v \in R$ and enumerate all rooms of $\mathcal{O}$, except R, that contain the wall $W = R \setminus \{v\}$. By definition of an oik, there is an odd number $k(W) - 1$ of such rooms. We get all rooms of $\mathcal{O}$ adjacent to R just repeating the above procedure for all $v \in R$.

Furthermore, by this procedure, it is also easy to obtain all room-selections adjacent to a given one $\mathcal{R} = \{R_1, \dots, R_k\}$ in a given oik-family $\mathcal{O} = \{\mathcal{O}_1, \dots, \mathcal{O}_k\}$.

The above definitions and observations immediately imply the following properties of the exchange graph $\mathcal{G}(\mathcal{V} \cup \mathcal{V}_1, \mathcal{E})$.

Lemma 1 *Vertices of $\mathcal{V}$ (room-partitions) and $\mathcal{V}_1$ (skew room-partitions) have odd and even degrees in $\mathcal{G}$, respectively. No two vertices of $\mathcal{V}$ are adjacent.* □

Obviously, the number $|\mathcal{V}|$ of the room-paritions is even, since in any graph number of the odd degree vertices is even.

Furthermore, given a room-partition $\mathcal{R} \in \mathcal{V}$, let us traverse $\mathcal{G}$ arbitrarily, yet, beginning in $\mathcal{R} \in \mathcal{V}$ and passing no edge twice, until no possible move is left. In other words, we construct an Eulerian path (a connected subgraph with two odd degree vertices and the rest of even degree) beginning in an odd degree vertex (room-partition) $\mathcal{R} \in \mathcal{V}$. Obviously, walking along any such path ends at another odd degree vertex (room-partition) $\mathcal{R}' \in \mathcal{V}$ distinct from $\mathcal{R}$. Indeed, $\mathcal{R}' \notin \mathcal{V}_1$, since all vertices of $\mathcal{V}_1$ have even degrees. Also $\mathcal{R}' \neq \mathcal{R}$, since vertex $\mathcal{R} \in \mathcal{V}$ is of odd degree. In particular, the following statement follows.

Theorem 1 *[1] Every oik-family $\mathcal{O}$ has an even number of room-partitions.*

Given a vertex $w \in V$ and a room-partition $\mathcal{R}$, we get another room-partition $\mathcal{R}'$ distinct from $\mathcal{R}$ by traversing the exchange graph $\mathcal{G}(\mathcal{O}, w)$ starting in $\mathcal{R}$ and passing no edge twice. □

If $\mathcal{O}$ is a family of 2-adjacent oiks then obviously vertices of $\mathcal{V}$ and $\mathcal{V}_1$ have degrees 1 and 2 respectively. In this case the exchange graph has a very simple structure: it is a disjoint union of simple paths whose ends form $\mathcal{V}$ and simple cycles whose vertices form the rest of $\mathcal{V}_1$. These paths uniquely define the traversing procedure, as well as a matching on the set $\mathcal{V}$ of room-partitions.

The above results can be generalized in many ways; for example, as follows.

Let $\delta : V \to \mathbb{Z}_+$ be a mapping of V into set $\mathbb{Z}_+$ of the non-negative integers. A room-selection $\mathcal{R}$ is called a δ-selection if $\deg_{\mathcal{R}}(v) = \delta(v)$ for each $v \in V$.

Given $\mathcal{O}$ and δ, let us define $\mathcal{V}$ as the set of all δ-selections and $\mathcal{V}_1$ as follows.

Let us fix a vertex $w \in W$ such that $\delta(w)$ is **odd**. A *skew* $(\delta \pm 1)$-selection (or a dragonfly) is defined as a room-selection $\mathcal{R}'$ such that $\deg_{\mathcal{R}'}(w) = \delta(w) - 1$, there is a vertex $u \in V$ such that $\deg_{\mathcal{R}'}(u) = \delta(u) + 1$, and $\deg_{\mathcal{R}'}(v) = \delta(v)$ for all other vertices $v \in V \setminus \{u, w\}$.

Given w and δ, let $\mathcal{V}_1$ be the set of all *skew* $(\delta \pm 1)$-selections. Finally, the adjacency relation $\mathcal{E}$ on the vertex-set $\mathcal{V} \cup \mathcal{V}_1$ and the exchange graph $\mathcal{G} = \mathcal{G}(\mathcal{O}, w) = \mathcal{G}(\mathcal{V} \cup \mathcal{V}_1, \mathcal{E})$ are defined exactly as before. It is easy to verify that all claims of Lemma 1 and Theorem 1 still hold.

Lemma 2 *Vertices of $\mathcal{V}$ (δ-selections) and $\mathcal{V}_1$ (skew $(\delta \pm 1)$-selection) have odd and even degrees in $\mathcal{G}$, respectively. No two vertices of $\mathcal{V}$ are adjacent.* □

Theorem 2 *[1] Every oik-family $\mathcal{O}$ has an even number of δ-selections. Given a vertex $w \in V$ of* **odd** *$\delta(w)$ and a δ-selection $\mathcal{R}$, we get another room-partition $\mathcal{R}'$ distinct from $\mathcal{R}$ by traversing the exchange graph $\mathcal{G}(\mathcal{O}, w)$ starting in $\mathcal{R}$ and passing no edge twice.* □

1.3 Main applications of oiks

Several classical results can be explained in terms of oiks and exchange algorithms, which, given a room partition find another one.

The Lemke-Howson algorithm [8] (of finding a Nash equilibrium in mixed strategies in a bimatrix game) can be interpreted as the exchange algorithm for two polytopal oiks.

The famous Sperner Lemma can be interpreted as Lemma 1 and Theorem 1 for an oik-family which consists of two oiks: a polytopal and Sperner one. In this case, given a multi-colored simplicial facet of a polytope, the exchange algorithm finds another one.

2 Every oik-family can be reduced to a pair of oiks one of which is a Sperner oik

Given a d-dimensional polytope (or, more generally, a $(d-1)$-dimensional manifold) P whose n vertices are colored by d colors $[d] = \{1, \ldots, d\}$, we also assume that P is *simplicial*, that is, every facet of P contains only d vertices. A facet is called *multi-colored* if its d vertices are colored by d distinct colors. Our version of Sperner's Lemma claims that the number of the multi-colored facets is even; moreover, given one of them, another one is uniquely determined by the exchange algorithm.

Let $\mathcal{O}_1 = (V, \mathcal{R}_1)$ be an oik whose n vertices are colored by d colors, $c : V \to [d]$. A room $R_1 \in \mathcal{R}_1$ is *multi-colored* if $c(R_1) = [d]$. By Theorem 1, the number of multi-colored rooms is even; moreover, given one of them, another one can be obtained by the exchange algorithm.

To see that Theorem 1 is applicable, let us add to the oik $\mathcal{O}_1$ a $(n-d)$-dimensional Sperner oik $\mathcal{O}_2 = (V, \mathcal{R}_2)$ defined on the same vertex-set V by the coloring c as follows. A set $R_2 \subseteq V$ is a room of oik $\mathcal{O}_2$ if and only if $|R_2| = n - d$ and the **complementary** set $V \setminus R_2$ of cardinality d is multi-colored; see Example 4. By this definition, a room $R_1 \in \mathcal{R}_1$ is multi-colored in oik $\mathcal{O}_1$ if and only if its complement $R_2 = V \setminus R_1$ is a room of $\mathcal{O}_2$, or in other words, sets R_1 and R_2 form a room-partition in the oik-pair $\mathcal{O} = (\mathcal{O}_1, \mathcal{O}_2)$.

Thus, Theorem 1 is applicable; in particular, it results in the standard "geometrical" Sperner Lemmas when $\mathcal{O}_1$ is a PM-, M-, or P-oik; see Example 1. Yet, in general, this approach is purely combinatorial and geometry is ignored. Moreover, oik $\mathcal{O}_1$ might be not 2-adjacent. In this case, given a room-partition, another one, defined in Theorem 1, is not necessarily unique.

Now let $\mathcal{O} = (\mathcal{O}_1, \mathcal{O}_2)$ be an arbitrary oik-pair defined on a common vertex-set. Then, Theorem 1 results in the

(a) Sperner Lemma when $\mathcal{O}_1$ is a polytopal oik, while $\mathcal{O}_2$ is a Sperner oik;

(b) Scarf Theorem [11] when $\mathcal{O}_1$ is a polytopal oik, while $\mathcal{O}_2$ is a Scarf oik;

(c) Lemke-Howson algorithm [8] when the oiks $\mathcal{O}_1$ and $\mathcal{O}_2$ are polytopal.

Somewhat surprisingly, an arbitrary oik-family $\mathcal{O} = (\mathcal{O}_1, \ldots, \mathcal{O}_k)$ is equivalent with an oik-pair $\mathcal{O}' = (\mathcal{O}_{k+1}, \mathcal{O}_0)$, where $\mathcal{O}_{k+1} = \mathcal{O}_1 + \cdots + \mathcal{O}_k$ is a sum, which will be defined below, and $\mathcal{O}_0$ is a Sperner oik, that is, the exchange graphs of $\mathcal{O}$ and $\mathcal{O}'$ are isomorphic. Hence, one can execute the exchange algorithm for $\mathcal{O}'$ rather than for $\mathcal{O}$.

Remark 4 *In particular, due to this reduction, the Scarf Theorem [11] can be derived from the Sperner Lemma as well as from the Scarf Lemma. The last observation is the main result of the recent paper by Kiraly and Pap [6].*

The reduction is simple. Let $\mathcal{O} = (\mathcal{O}_1, \ldots, \mathcal{O}_k)$ be an arbitrary oik-family in which $\mathcal{O}_i = (V, \mathcal{R}_i)$ is a d_i-dimensional oik for $i \in [k] = \{1, \ldots, k\}$ and $\sum_{i=1}^k d_i = n = |V|$. First, let us define the sum $\mathcal{O}_{k+1} = \sum_{i=1}^k \mathcal{O}_i$ as follows: $\mathcal{O}_{k+1} = (kV, \mathcal{R}_{k+1})$, where kV consists of k pairwise disjoint copies $V_1, \ldots, V_k$ of V and $R \in \mathcal{R}_{k+1}$ if and only if $R \cap V_i$ is a room of the oik $\mathcal{O}_i = (V_i, \mathcal{R}_i)$ (which is a copy of $\mathcal{O}_i = (V, \mathcal{R}_i)$) for all $i \in [k] = \{1, \ldots, k\}$. In particular, $|kV| = kn$ and $d_{k+1} = \sum_{i=1}^k d_i = n$ are the size and dimension of the oik $\mathcal{O}_{k+1}$.

Let us color n vertices of V by n pairwise distinct colors and then copy this coloring in every V_i, $i \in [k]$, thus, coloring kn vertices of the set kV in n colors. This coloring standardly defines the Sperner oik $\mathcal{O}_0 = (kV, \mathcal{R}_0)$ in which $R \in \mathcal{R}_0$ if and only if $kV \setminus R$ is multi-colored. Thus, the oik-pair $\mathcal{O}' = (\mathcal{O}_{k+1}, \mathcal{O}')$ is defined. Let us choose two vertices: $w \in V$ and $w' \in kV$.

Theorem 3 *Two exchange graphs $\mathcal{G} = \mathcal{G}(\mathcal{O}, w)$ and $\mathcal{G}' = \mathcal{G}(\mathcal{O}', w')$ are isomorphic whenever vertices w and w' are of the same color.*

The proof can be found in [2].

It is important to notice that the obtained reduction is exponential in k but it is polynomial in size of $\mathcal{O}$. Hence, it is polynomial when k is a constant.

Thus, the room-partitions of an arbitrary oik-family $\mathcal{O} = \{\mathcal{O}_1, \ldots, \mathcal{O}_k\}$ are in one-to-one correspondence with the multi-colored rooms of the sum $\mathcal{O}_1 + \ldots + \mathcal{O}_k$. In particular, instead of looking for another room-partition, one can search for another multi-colored room. This shows a sort of universality of the Sperner Lemma, cf. [6].

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