Euler Complexes (Oiks)

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Abstract

We present a class of instances of the existence of a second object of a specified type, in fact, of an even number of objects of a specified type, which generalizes the existence of an equilibrium for bimatrix games. The proof is an abstract generalization of the Lemke-Howson algorithm for finding an equilibrium of a bimatrix game.

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A *d-oik*, $C = (V, F)$, short for *d-dimensional Euler complex*, $d \geq 1$, is a finite set V of elements called the *vertices* of C and a family of $(d+1)$-element subsets of V , called the *rooms* of C, such that every d-element subset of V is in an even number of the rooms. A *wall* of a room means a set obtained by deleting one vertex of the room - and so any wall of a room in an oik is the wall of a positive even number of rooms of the oik.

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Example 1. A d-dimensional simplicial pseudo-manifold is a d-oik where every d-element subset of vertices is in exactly zero or two rooms, i.e., in a simplicial pseudo-manifold any wall is the wall of exactly two rooms. Explicitly listed pseudo-manifold oiks, such as triangulations of a sphere, treated in [5], are important because algorithm efficiency is relative to number of vertices and number of rooms.

Example 2, Polytopal Oiks. Let $\{Ax = b, x \geq 0\}$, be a tableau as in the simplex method, whose solution-set is bounded and whose basic feasible solutions have all basic variables non-zero (non-degenerate). Let V be the column-set of A. Let the rooms be the subsets S of columns such that $V - S$ is a feasible basis of the tableau. This is an $(n - r - 1)$-oik where n is the number of columns of A and r is the rank (the number of rows) of A. It is combinatorially the boundary of a 'simplicial polytope'. The set of rooms of a polytopal oik, which is given implicitly by a tableau, is typically exponentially large relative to the size of the tableau, and so the efficiency of an algorithm is relative to size of the tableau rather than number of rooms.

Example 3, Sperner Oiks. Let the n members of set V be colored with r colors. Let the rooms be the subsets S of V such that $V - S$ contains exactly one vertex of each color. This is an $(n - r - 1)$-oik. In fact it is the oik of Example 2 where each column of A is all zeroes except for one positive entry.

Example 4. An Euler graph, that is a graph such that each of its nodes is in an even number of its edges (the rooms), is a 1-oik.

Example 5. For any connected Euler graph G with n vertices ($n \geq 3$), we have an $(n - 2)$-oik (V, K) where V is the set of edges of G and the rooms are the edge-sets of the spanning trees of G.

Example 6. For any connected bipartite graph G with m edges and n vertices we have an $(m - n)$-oik where V is the edge-set of G, and the rooms are the edge-complements of spanning trees of G.

Example 7, Gale Oiks. Where graph G is a simple cycle, let V be its set of nodes, and let a subset S of V be a member of F when the subgraph induced by S consists of paths each with an even number of nodes. Clearly $M = (V, F)$ is a pseudo-manifold oik. A *matching* in a graph G is a subset of its edges which includes any node of G at most once. Clearly, S is in F (i.e., is a room) if and only if it is the node-set uniquely of a matching in G. Such oiks are polytopal. Gale [6] discovered their well-known characterization as the combinatorial types of 'cyclic polytopes'. Algorithmic efficiency for Gale oik inputs is relative to the size G, rather than the number of rooms.

Let $M = [(V, F_i) : i = 1, ..., h]$ be an indexed collection of oiks (which we call an *oik-family*) all on the same vertex-set V . The oiks of M are not necessarily of the same dimension. Of course, they all may be the same oik.

A *room-family*, $R = [R_i : i = 1, ..., h]$, for oik-family M, is where, for each i , R_i is a room of oik i (i.e., a member of F_i). A room-partition R for M means a room-family whose rooms partition V , i.e., each vertex is in exactly one room of R.

For example, if the oiks are as in Example 4, and all the same, then a room-partition is the same as a matching which hits all of the nodes of G. The 'blossom method' is a well-known non-trivial polytime algorithm for finding a room partition of G or deciding there is none (see [9]). It can of course also be used in polynomial time to find a second room-partition or decide there is none.

If one of the oiks is as in Example 3, with an even number, r, of colors, and the other oiks are as in Example 4, and all the same, then a room-partition is the same as a matching in G whose nodes use each color once (see [3]). Clearly, by applying the blossom algorithm to the graph obtained from G by identifying nodes of the same color, we have a polytime algorithm for finding a room-partition, and second room partition, or their non-existence. In particular, for any Gale oik and for any coloring of its vertices, there is polytime algorithm for finding a first and second room of the Gale oik which each use each color once, or determining non-existence.

Theorem 1. *Given an oik-family M and a room-partition R for M, there exists another different room-partition for M. In fact, for any oik-family M, there is an even number of room-partitions.*

Proof. Choose a vertex, say w, to be special. A *w-skew room-family for oik-family M* means a room-family, $R = [R_i : i = 1, , h]$, for M such that w is not in any of the rooms R_i, some vertex v is in exactly two of the R_i , and every other vertex is in exactly one of the R_i .

Consider the so-called exchange-graph X, determined by M and w, where the nodes of X are all the room-partitions for M and all the w-skew room-families for M. Two nodes of X are joined by an edge of X if each is obtained from the other by replacing one room by another. It is easy to see that the odd-degree nodes of X are all the room-partitions for M, and all the even-degree nodes of X are the w-skew room-families for M. Hence there is an even number of room-partitions for M. □

'The Exchange algorithm': An algorithm for getting from one room-partition for M to another is to walk along a path in X, not repeating any edge of X, from one to another. Where each oik of the oik-family M is a simplicial pseudo-manifold, X consists of disjoint simple paths and simple cycles, and so the algorithm is uniquely determined by M and w.

The Lemke-Howson algorithm [7] for finding a Nash equilibrium of a bi-matrix game is this exchange algorithm applied to a pair of polytopal oiks.

Walter Morris [8] shows that for a certain sequence of polytopal pairs, where one is a Gale oik and the other is a Sperner oik, the length of the exchange-algorithm path grows exponentially. This provides another warning, of many, not to conclude from one approach to a problem not being polytime that there is not some other approach to the problem which is polytime.

Suppose each oik of an oik-family M is given by an explicit list of its rooms, each oik perhaps a simplicial pseudo-manifold, perhaps a 2-dimensional sphere. Is some path of the exchange graph not well-bounded by the number of rooms? The answer is in [5].

Theorem 2. (Sumset Oiks.) *Let $M = [(V, F_i) : i = 1, ..., h]$ be an oik-family. A disjoint room-family of M means a room-family $R = [R_i : i = 1, ..., h]$, where, for each i , R_i is a room of oik i (i.e., a member of F_i), and these rooms R_i are mutually disjoint.*

A sumset of M means the union of the members of a disjoint room-family of M. Where each sumset of M is smaller than V, let F be the set of sumsets of M.

Then (V, F) is an oik, called the sumset oik of M. If each of the oiks, (V, F_i), is a pseudo-manifold, then (V, F) is a pseudo-manifold. □

For example, a Gale oik is the sumset oik of several copies of a graph, G, which is a simple cycle, i.e., a 1-dimensional manifold, where the rooms are the edges of G.

It is an easy matter to describe an exponentially growing sequence of exchange graphs, X, each of them for finding a second properly colored room of a Gale oik, say $G(j)$, obtained by "gluing $G(j-1)$ into G".

Example 8. A pure $(d+1)$-complex, $C = (V, F)$, means simply a finite set, V , and a family, F, of $(d+2)$-element subsets. The boundary, $bd(C) = (V, bd(F))$, of any pure $(d+1)$-complex, C, means the pure d- complex where $bd(F)$ is the family of those $(d+1)$-element subsets of V which are subsets of an odd number of members of F.

For any pure $(d+1)$-complex, C, its boundary, $bd(C)$, is a d-oik. This is more-or-less the first theorem of simplicial homology theory. By recalling the meaning of d-oik, it is saying that for any pure $(d+1)$-complex, C, every d-element subset, H, of V is a subset of an even number of $(d+1)$-element sets which are subsets of an odd number of the $(d+2)$-element members of F. It can be proved graph-theoretically by observing that, for any d-element subset, H, of V , the following graph, G, has an even number of odd degree nodes: The nodes of G are the $(d+1)$-element subsets of V which contain H. Two of these $(d+1)$-element nodes are joined by an edge in G when their union is a $(d+2)$-element member of F. Clearly a node of G is a subset of an odd number of members of F, and hence is a member of $bd(F)$, when it is an odd-degree node of G. What can we say about $bd(F)$, besides Theorem 1, when F is the set of bases of a matroid?

In [2], different exchange graphs were studied. In [1], it was shown that Thomason's [10] exchange graph algorithm for finding a second hamiltonian circuit in a cubic graph is exponential relative to the size of the given graph.

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