Lehman's Game:

Given any system, Ax = b, over any field,
where column b is not all 0's.
“Cut” and “Short” take turns, cut going first.

Cut, in each of his turns, can delete a column of A
(in other words, set equal to 0 the corresponding variable of x).
He wins if he succeeds in deleting enough columns
(which then can not be secured by the short player)
so that there is no solution to Ax = b
using only secured columns.

Short, in each of his turns,
can secure any undeleted column of A
from ever being deleted.
He wins if he succeeds in getting a solution to Ax = b
using only secured columns.

A subset R of columns is said to **span** another set T of columns if each member of T is a linear combination of R.

To **contract** a column j of A means to do row operations on Ax = b so that column j has a non-zero in only one row, and then delete that one row and delete column j, to get a remaining system A'x' = b'.
As soon as a solution x' to A'x' = b' is found, it can be plugged into the deleted equation in order to determine a solution x of Ax = b.

The way Short can secure a column j from deletion by Cut is to contract column j.

Short Theorem. Short can be sure of winning if and only if **(5.1)** there are two disjoint linearly independent subsets, R and T, of the columns of A which each span column b, and are such that R and T are "cospanning", (meaning R is spanned by T, and T is spanned by R).

The following is a strategy for Short to be sure of winning by using the structure in (5.1):
whenever Cut deletes a column from R or T, say R, then Short contracts a column of T to get a system A'x' = b' in which the remaining R' and T' satisfy (5.1).

Cut Theorem. Cut can be sure of winning if and only if
(5.2) there is a “contraction” of a subset S of the columns of A so that in the resulting system, say A'x' = b', the columns of A' can be partioned into two linearly independent sets, J and K, such that at least one of them does not span b'.

Cuts first move is to delete a column so that
what remains of J and K is such that neither spans b'.

The following is a strategy for Cut to be sure winning
by using the structure in (5.2).
Whenever Short has played to get a system A'x' = b'
in which A' partitions into two independent sets, J' and K',
where at most one of them spans b',
Cut deletes a column so that what remains of J' and K'
is such that neither spans b'.

The strategy for Short to be sure of winning
by using structure (5.1), and
the strategy for cut to be sure winning
by using the structure in (5.2),
proves the "if" part of each theorem,
and hence also proves that
not both of (5.1) and (5.2) can hold.

To prove the "only if" part of both of the theorems
we give a polytime algorithm for finding
either an instance of (5.1) or an instance of (5.2).

The methods used are presented
in the A&PT lecture on **Polymatroids**,
Wednesday, Feb. 10 at 10 a.m.

The Shannon Switching Game
is an elegant special case of Lehman's game
which is played on an undirected graph G with a specified edge, b.

The cut player in each of his turns may delete one edge,
different from b.
The short player in each of his turns may secure one edge, e,
different from b, against being deleted
(e.g., by “shrinking” edge e to a node).

Short wins if he succeeds in securing a set S of edges which
includes the edges of a path between the two nodes of b
(e.g., by shrinking a set S of edges
so that the two ends of b become a single node).
Cut wins if he succeeds in deleting edges
which (together with b) separate the two nodes of b.

This graph version of Lehman's game is modeled by the system,
Ax = b, over the field where 1 + 1 = 0,
by letting the rows of A be the nodes of G,
letting the columns of A be the edges of G,
and letting an entry in A be 1 when that edge hits that node.
Find the simple interpretations of the two theorems
for this graph version.

Hamidoune and Las Vergnas develop a theory of
"Directed Switching Games on Graphs and Matroids"
in Journal of Combinatorial Theory, B, 40, 237-269 (1986).
(Question:
To what extent can it be seen as instances of Lehman's game?)

**Where this abstract is posted on the website for
the didactic seminar, "Algorithms and Pretty Theorems",
it will link to detailed materials
on the games discussed here.**