

## Polymatroids, Part 2

Polymatroids were first motivated by an interest in unifying the earliest result in combinatorial optimization, Egervary 1931, the Hungarian method, with the next earliest result in combinatorial optimization, Boruvka 1934, the "greedy algorithm" for finding a minimum cost spanning tree in a graph.

Since Boruvka's algorithm is equivalent to minimizing a linear function over the incidence vectors of the edge-sets of forests in a graph, it is the same as maximizing over the convex hull of those vectors.

That prompted the question, "what is it about the set of inequalities defining that hull which makes the greedy algorithm work?".

The answer is that the hull be of the form  $P(f)$  as in (3.4). In other words, the answer is that the inequalities defining the hull be of the form  $x \geq 0$ , and  $x(S) \leq f(S)$  for every subset  $S$  of  $E$ , where  $f$  is a polymatroid function.

The **linear programming dual** of  
 maximize  $\{cx: x \geq 0, \text{ and } x(S) \leq f(S)\}$   
 is the linear program,  
 minimize  $yf \equiv \sum [f(S)y(S): \text{ subsets } S \text{ of } E]$ ,  
 where  $y$  satisfies  $y \geq 0$  and,  
 for every  $j \in E, \sum [y(S) : j \in S] \geq c(j)$ .

The obvious weak lp duality inequality says that  $cx \leq yf$ , and so, if we find a feasible  $x$  and a feasible  $y$  such that  $cx = yf$ , then we know that the  $x$  and  $y$  are optimum.

The following theorem gives "the greedy algorithm", and a "dual greedy algorithm", for any polymatroid,  $P(f)$ , as given by in (3.3).

Consider that the elements of  $E$  are ordered  $1, 2, 3, \dots$ , so that  $c(1) \geq c(2) \geq \dots \geq c(k) > 0 \geq c(k+1) \geq \dots$   
 For each integer  $i = 1, 2, \dots, k$ , let  $S(i) \equiv$  the set  $\{1, \dots, i\}$ .

(3.9) Theorem (**The Greedy Algorithm**).

$cx$  is maximized over  $x \in P(f)$  by the following vector  $x$ :  
 $x(1) = f(S(1))$ ;  
 $x(i) = f(S(i)) - f(S(i-1))$  for  $i = 2, \dots, k$ ;  
 and  $x(i) = 0$  for  $i > k$ .

(3.10) Theorem (**The Dual Greedy Algorithm**).

$yf$  is minimized for the dual lp by the following vector  $y$ :  
 $y(S(i)) = c(i) - c(i + 1)$  for  $i = 1, \dots, k-1$ ;  
 $y(S(k)) = c(k)$ ; and  $y(S) = 0$  for every other subset of  $E$ .

Clearly,  $cx = yf$  for this  $x$  and this  $y$ . Clearly this  $y$  is dual feasible. Hence, the theorems are proved by using the definition of  $f$  being a polymatroid function to prove that the  $x$  of (3.9) is in  $P(f)$ .

(3.11) Corollary of (3.9). **The vertices of  $P(f)$  are** the vectors of the form in (3.9) for the various orderings,  $1, \dots, k$ , of subsets of  $E$ .

In particular, for integer valued  $f$  they are integer valued.

Where  $f$  is the rank function of a matroid, the vertices of  $P(f)$  are the 0,1 incidence vectors of the independent sets.

The first part of the corollary follows because a point  $x$  is a vertex of a polyhedron  $P$  if and only if some linear function is uniquely maximized over  $P$  by  $x$ .

The last part of the corollary follows because, where  $f$  is the rank function of a matroid,  $x(i) = f(S(i)) - f(S(i-1))$  is 1 when element  $i$  together with any basis of  $S(i-1)$  is a basis of  $S(i)$ , and is 0 when element  $i$  together with any basis of  $S(i-1)$  is dependent.

Where  $f$  is the rank function of a matroid  $(E, F)$ , (3.9) is saying that you get an independent set  $J$  with the largest sum of  $c$ -values of its elements

by choosing at each step to put into  $J$  the element having largest non-negative  $c$ -value among the elements which would maintain the independence of  $J$ .

In other words, with the elements of  $E$  ordered according to non-increasing  $c$ -values, you get an optimum  $J$

by deciding in that order whether to insert an element into  $J$ .

A most amazing and useful theorem of polymatroid theory is:

**(3.12) The Polymatroid Intersection Theorem.**

For any two integer-valued polymatroid functions, say  $f$  and  $g$ , of subsets of the same ground set  $E$ , and for any integer-valued  $a$ ,  $b$ , and  $c$ , each coordinatized by  $E$ , the maximum of  $cx$  over  $x$  satisfying  $a \leq x \leq b$  and  $x \in P(f) \cap P(g)$  is achieved by an integer-valued  $x$ .

And the minimum for the dual of this lp is also achieved by integers.

There are slick proofs, and there is a polytime algorithm for finding a maximizing  $x$ , and a minimizing dual solution

(3.13) One amazing corollary is that

For the polytopes,  $P(f)$  and  $P(g)$ , of any two matroids on ground-set  $E$ , the vertices of  $P(f) \cap P(g)$  are exactly the vectors of the subsets of  $E$  which are independent in both matroids.

In other words the vertex-set of  $P(f) \cap P(g)$  equals the vertex-set of  $P(f)$  intersected with the vertex-set of  $P(g)$ . In general, the vertices of the intersection of three matroid polytopes on ground-set  $E$  are fractional garbage.

Where  $c$  is all ones,  $a$  is all 0's, and  $b$  is all anything too big to be active, we have the best known instance of the intersection theorem:

(3.12) For any two matroids on the same ground-set  $E$ , with rank functions  $f$  and  $g$  respectively, the maximum size of a set independent in both matroids = minimum of  $[f(S) + g(E - S)]$  over all subsets  $S$  of  $E$ .

We will see an immediate proof of (3.8) using (3.12).