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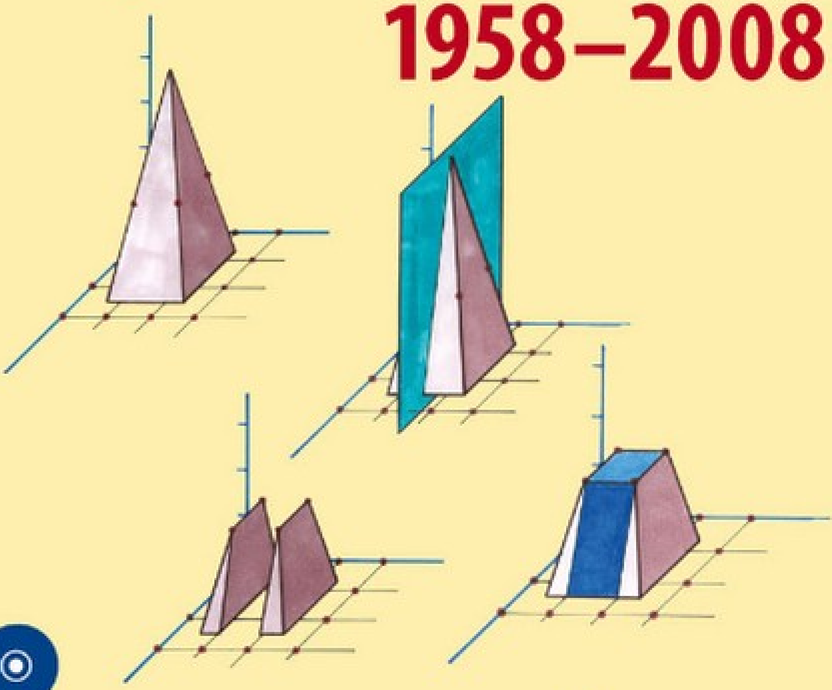
50 Years of Integer Programming 1958–2008



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Chapter 7
Matroid Partition

Jack Edmonds

Introduction by *Jack Edmonds*

This article, “Matroid Partition”, which first appeared in the book edited by George Dantzig and Pete Veinott, is important to me for many reasons: First for personal memories of my mentors, Alan J. Goldman, George Dantzig, and Al Tucker. Second, for memories of close friends, as well as mentors, Al Lehman, Ray Fulkerson, and Alan Hoffman. Third, for memories of Pete Veinott, who, many years after he invited and published the present paper, became a closest friend. And, finally, for memories of how my mixed-blessing obsession with good characterizations and good algorithms developed.

Alan Goldman was my boss at the National Bureau of Standards in Washington, D.C., now the National Institutes of Science and Technology, in the suburbs. He meticulously vetted all of my math including this paper, and I would not have been a math researcher at all if he had not encouraged it when I was a university drop-out trying to support a baby and stay-at-home teenage wife. His mentor at Princeton, Al Tucker, through him of course, invited me with my child and wife to be one of the three junior participants in a 1963 Summer of Combinatorics at the Rand Corporation in California, across the road from Muscle Beach. The Bureau chiefs would not approve this so I quit my job at the Bureau so that I could attend. At the end of the summer Alan hired me back with a big raise.

Dantzig was and still is the only historically towering person I have known. He cared about me from a few days before my preaching at Rand about blossoms and about good algorithms and good characterizations. There were some eminent combinatorial hecklers at my presentation but support from Dantzig, and Alan Hoffman, made me brave.

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I think of Bertrand Russell, Alan Turing, and George Dantzig as the three most important philosophers of the last century. During an infrequent visit to California from Washington, D.C., sometime in the 60s, Dantzig took me, a wife, and three kids, to Marineland and also to see a new shopping mall in order to prove to us that having a ceiling of a certain height in his carefully planned Compact City is as good as a sky.

One time when I unexpectedly dropped in on Dantzig, the thrill of my life was him asking me to lecture to his linear programming class about how the number of pivots of a simplex method can grow exponentially for non-degenerate linear programming formulations of shortest path problems, and also asking me to vet contributions for a math programming symposium which he was organizing.

One of my great joys with George Dantzig was when a friend working at Hewlett-Packard asked me to come discuss the future of operations research with his artificial intelligence colleagues. I was discouraged when no one I knew in O.R. seemed interested in helping—that is, until I asked George. He told my second wife Kathie and me that he was a neighbor and had socialized with Mr. Hewlett, or was it Mr. Packard, for years, and had never been invited to HP, two blocks away. George took over the show and was wonderful. Kathie video-taped it. The next morning he asked if she had made him a copy yet.

Al Tucker made me a Research Associate and put me in charge of his Combinatorics Seminar at Princeton during 1963–64. Combinatorists whom I wanted to meet accepted paying their own way to speak at my ‘Princeton Combinatorics and Games Seminar’. However, except for Ron Graham who came over from Bell, and Moses Richardson who came down from City University, they were unable to schedule their visits. So I hastily organized a Princeton Conference in the spring of 1964 where the eminent seminar invitees could lecture to each other.

At that conference I met Al Lehman who led me, by his matroidal treatment of what he called the Shannon switching game, to see that matroids are important for oracle-based good algorithms and characterizations. I persuaded Al, along with Chris Witzgall, to come work at the Bureau of Standards, and immediately we started looking for people to participate in a two-week Matroid Workshop at the Bureau of Standards in autumn 1964. We didn’t find more than six who had even heard of the term ‘matroid’. About twenty serious people came to it, including Ray Fulkerson, George Minty, Henry Crapo, Dan Younger, Neil Robertson, and Bill Tutte. Within a year it seemed the whole world was discovering matroids.

The Bureau was delighted at the prospect of hiring Al Lehman. However, an aftermath of McCartheism left the Bureau with the rule that new employees had to take an oath of loyalty. The early computer-guru, Ida Rhodes, actually tugged at Al’s arm to try to get him to take the oath but he wouldn’t. Instead he took a research job with a Johns Hopkins satellite of the U.S. Army which did not require such an oath. He literally picketed the Matroid Workshop, speaking to whomever would listen about the ‘Bureau of Double Standards’. We stayed friends for the many years until his recent death in Toronto.

At the same workshop, Gian-Carlo Rota conceived of and started organizing the Journal of Combinatorial Theory. He also insisted that the ‘ineffably cacophonic word matroid’ be replaced by ‘combinatorial geometry’.

George Minty was an especially sweet and brilliant participant. He wrote a paper which Bob Bland credits with being a precursor of oriented matroids. He spent years afterwards on successfully extending the good algorithm for optimum matchings in a graph to optimum independent sets in a clawfree graph. His work is still the most interesting aspect of matching theory.

During the year after the Matroid Workshop, Ray Fulkerson and I regularly spent hours talking math by government telephone between Santa Monica and Washington. Ray and I never did learn how to work computers, and though I think the prototype of email did exist back then in our government circles, he and I didn’t know about it. One of the outcomes of our talk was combining a version of the matroid partitioning algorithm described in the paper here with Ray’s interest in doing everything possible by using network flow methods.

My huff about him and Ellis Johnson calling the blossom method “a primal-dual method” led me to look for algorithms for network flow problems which were polytime relative to the number of bits in the capacities as well as in the costs. The reason I had presented the blossom method only for 1-matchings is that for b-matchings I could not call it a “good algorithm” until I had figured out how to do that for network flows. Once it’s done for flows, it’s easy to reduce optimum b-matchings to a flow problem and a b-matching problem where the b is ones and twos. Dick Karp was independently developing good algorithms for network flows and so much later I published with Dick instead of, as intended, with Ray and Ellis. I enjoyed working with Ray and I coined the terms “clutter” and “blocker”. I can’t remember who suggested the term “greedy” but it must have been Alan Goldman and probably Ray as well.

It was important to me to ask Ray to check with the subadditive set function expert he knew about submodular set functions. When the answer came back that they are probably the same as convex functions of additive set functions, I knew I had a new tiger by the tail.

Ray and I liked to show off to each other. I bragged to him about discovering the disjoint branchings theorem, mentioned later. Trouble is, I then became desperate to find quickly a correction of my faulty proof. I think I would have done a better job on the theorem if I had not been frantic to cover my hubris.

During a phone call, Ray mentioned that one day later, four months after the Matroid Workshop, there would be a combinatorics workshop in Waterloo. My boss Alan Goldman rescued me as usual and I quickly hopped a plane to Canada to sleep along with George Minty on sofas in Tutte’s living room.

Neil Robertson, a meticulous note-taker, had reported to Crispin Nash-Williams on my Matroid Workshop lectures. Crispin, by his own description, was too enthusiastic about them. He was giving a keynote lecture about matroid partitioning on the first morning of this Waterloo workshop. I felt compelled immediately after his talk to speak for an impromptu hour on the following:

Theorem 1. A non-negative, monotone, submodular set function, $f(S)$, of the subsets S of a finite set E, is called a polymatroid function on E. For any integer-valued polymatroid function on E, let F be the family of subsets J of E such that for every non-empty subset S of J, the cardinality of S is at most $f(S)$. Then $M = (E,F)$ is a matroid. Its rank function is, for every subset A of E, $r(A)$, meaning max[cardinality of a subset of A which is in F] = min[$f(S)$ + cardinality of $(A \setminus S)$ for any subset S of A].

After this opening of the Waterloo meeting I urgently needed a mimeographed abstract handout and so I submitted Theorem 1.

The theorem is dramatic because people had only seen matroids as an axiomatic abstraction of algebraic independence, and not as something so concrete as a kind of linear programming construction quite different from algebraic independence.

I tried to explain on that snowy April Waterloo morning how the theorem is a corollary of a theory of a class of polyhedra, called polymatroids, given by non-negative vectors x satisfying inequality systems of the form:

For every subset S of E, the sum of the coordinates of x indexed by the j in S is at most $f(S)$.

However, even now, this is often outside the interest of graph theorists, or formal axiomatists. I am sorry when expositions of matroid theory still treat the subject only as axiomatic abstract algebra, citing the mimeographed abstract of that Waterloo meeting with no hint about the linear programming foundations of pure matroid theory.

What does Theorem 1 have to do with matroid partitioning? Well—the rank function of a matroid is a polymatroid function, and hence so is the sum of the rank functions of any family of matroids all on the same set E. Hence a special case of Theorem 1, applied to this sum, yields a matroid on E as the 'sum' of matroids on E. I had hoped to understand the prime matroids relative to this sum, but, so far, not much has come of that.

Suppose we have an oracle which for an integer polymatroid function $f(S)$ on E gives the value of $f(S)$ for any subset S of E. Then the theorem gives an easy way to recognize when a given subset J of E is not a member of F, in other words not independent in the matroid determined by Theorem 1. Simply observe some single subset S of J having cardinality greater than $f(S)$.

Does there exist an easy way to recognize when a set J is independent? The answer is yes. For a general integer polymatroid function f, this easy way needs some of the linear programming theory which led me to Theorem 1, which I will describe in a moment.

However for the special case of Theorem 1 where f is the sum of a given family, say H, of matroid rank functions, an easy way to recognize that a set J is independent, which even the most lp resistant combinatorist can appreciate, is given by the 'matroid partition theorem' of the present paper: a set J is independent if and only if it can be partitioned into a family of sets, which correspond to the members of H, and which are independent respectively in the matroids of H.

Thus, relative to oracles for the matroids of H, for the matroid M determined as in Theorem 1 by the f which is the sum of the rank functions of H, we have a 'good characterization' for whether or not a subset J of E is independent in M. To me this meant that there was an excellent chance of proving the matroid partition theorem by a good algorithm which, for a given J, decides whether or not J is independent in matroid M. That is what the present paper does.

Having an instance of a good characterization relative to an oracle, and having a good algorithm relative to the oracle which proves the good characterization, was the main point and motivation for the subject.

One reason I like the choice of "Matroid Partition" for the present volume is that, as far as I know, it is the first time that the idea of what is now called NP explicitly appears in mathematics. The idea of NP is what forced me to try to do some mathematics, and it has been my obsession since 1962.

I talked about it with Knuth at about that time and ten years later he asked me to vote on whether to call it NP. I regret that I did not respond. I did not see what non-deterministic had to do with it. NP is a very positive thing and it has saddened me for these many years that the justified success of the theory of NP-completeness has so often been interpreted as giving a bad rap to NP.

Let me turn my attention to linear programming which gave me Theorem 1, which led to the present paper.

Given the enormous success that the marriage problem and network flows had had with linear programming, I wanted to understand the goodness of optimum spanning trees in the context of linear programming. I wanted to find some combinatorial example of linear programming duality which was not an optimum network flow problem. Until optimum matchings, every min max theorem in combinatorics which was understood to be linear programming was in fact derivable from network flows—thanks in great measure to Alan Hoffman and Ray Fulkerson. Since that was (slightly) before my time, I took it for granted as ancient.

It seemed to be more or less presumed that the goodness of network flow came from the fact that an optimum flow problem could be written explicitly as a linear program. The Farkas lemma and the duality theorem of linear programming are good characterizations for explicitly written linear programs. It occurred to me, preceding any success with the idea, that if you know a polytope as the hull of a set of points with a good, i.e., easily recognizable, description, and you also know that polytope as the solution-set of a set of inequalities with a good description, then using lp duality you have a good characterization. And I hoped, and still hope, that if you have good characterization then there exists a good algorithm which proves it. This philosophy worked for optimum matchings. It eventually worked for explicitly written linear programs. I hoped in looking at spanning trees, and I still hope, that it works in many other contexts.

The main thing I learned about matroids from my forefathers, other than Lehman, is that the edge-sets of forests in a graph are the independent sets of a matroid, called the matroid of the graph. What is it about a matroid which could be relevant to a set of linear inequalities determining the polytope which is the hull of the 0-1 vectors of independent sets of the matroid? The rank function of course. Well what is it about the rank function of a matroid which makes that polytope extraordinarily nice for optimizing over? That it is a polymatroid function of course. So we're on our way to being pure matroid theorists.

A "polymatroid" is the polytope $P(f)$ of non-negative solutions to the system of inequalities where the vectors of coefficients of the vector of variables is the 0-1 vectors of subsets S of E and the r.h.s. constants are the values of the polymatroidal function $f(S)$. It turns out that it is as easy, relative to an oracle for f, to optimimize any linear function over $P(f)$, as it is to find a maximum weight forest in an edge-weighted graph. Hence it is easy to describe a set of points for which $P(f)$ is the convex hull. Where f is the rank function of a matroid, those points are the 0-1 vectors of the independent sets of the matroid, in particular of the edge-sets of the forests for the matroid of a graph.

A polymatroid has other nice properties. For example, one especially relevant here is that any polymatroid intersected with any box, $0 \leq x \leq a$, is a polymatroid. In particular, any integer-valued polymatroid function gives a polymatroid which intersected with a unit cube, $0 \leq x \leq 1$, is the polytope of a matroid. That is Theorem 1.

So what? Is this linear programming needed to understand Theorem 1? Not to prove it, though it helps. For Theorem 1, rather than for any box, the lp proof can be specialized, though not simplified, to being more elementary. However linear programming helps answer "yes" to the crucial question asked earlier: Does there exist an easy way to recognize when a set J is independent?

It is obvious that the 0-1 vector of the set J is in the unit box. Using the oracle for function f we can easily recognize if J is not independent by seeing just one of the inequalities defining $P(f)$ violated by the 0-1 vector of J. But if the vector of J satisfies all of those inequalities, and hence J is independent in the matroid M described by Theorem 1, how can we recognize that? Well using linear programming theory you can immediately answer. We have mentioned that we have a very easy algorithm for optimizing over polytope $P(f)$ and so, where n is the size of the ground set E which indexes the coordinates of the points of $P(f)$, we have an easy way to recognize any size $n+1$ subset of points each of which optimizes some linear objective over $P(f)$. Linear programming theory tells that the 0-1 vector of J is in $P(f)$, and hence J is independent, if and only if it is a convex combination of some $n+1$ points each of which optimizes some linear function over $P(f)$.

That's it. We have a good characterization of whether or not a set J is independent in the matroid described by Theorem 1. It takes a lot more work to say that directly without linear programming. We do that in the paper here with the matroid partition theorem for the case where f is the sum of some given matroid rank functions.

For concreteness assume that a is any vector of non-negative integers corresponding to the elements of finite set E. Of course Theorem 1 is the special case for a unit box of the theorem which says that box, $0 \leq x \leq a$, intersected with integer polymatroid, $P(f)$, is an integer polymatroid. Call it $P(f,a)$.

The rank $r(f,a)$ of $P(f,a)$, meaning the maximum sum of coordinates of an integer valued vector x in $P(f,a)$ is equal to the minimum of $f(S)$ + the sum of the coordinates of a which correspond to $E \setminus S$. If you know the meaning of a submodular set function, the proof of this is very easy. At the same time, you prove that the max sum of x is achieved by taking any integer valued x in $P(f,a)$, such as the zero vector, and pushing up the value of its coordinates in any way you can while staying in $P(f,a)$. (By analogy with matroid, having this property is in fact the way we define polymatroid.) The only difficulty with this otherwise easy algorithm is deciding how to be sure that the x stays in $P(f,a)$. Hence the crux of the problem algorithmically is getting an algorithm for deciding whether or not a given x is in $P(f,a)$. We do get a good characterization for recognizing whether or not an x is in $P(f,a)$ in the same way we suggested for characterizing whether or not a subset J of E is member of matroid M. Hence from this we have a good characterization of the rank $r(f,a)$ without necessarily having a good algorithm for determining the rank $r(f,a)$.

Any integer-valued submodular set function $g(S)$, not necessarily monotone or non-negative, can be easily represented in the form constant + $f(S)$ + the sum of the coordinates of a which correspond to $E \setminus S$, where f is an integer polymatroid function and a is a vector of non-negative integers. Hence, since the mid sixties, we have had a good characterization of the minimum of a general integer-valued submodular function, relative to an oracle for evaluating it. Lovász expressed to me a strong interest in finding a good algorithm for it in the early seventies. He, Grötschel, and Schrijver, showed in the late seventies that the ellipsoid method for linear programming officially provides such an algorithm. However it has taken many years, many papers, and the efforts of many people, to get satisfying direct algorithms, and this currently still has wide research interest. We have observed here how the matroid partitioning algorithm was a first step. The methods by which Dick Karp and I got algorithms for network flows was another first step.

There are other interesting things to say about matroid and submodular set-function optimization theory which I won't mention, but there is one I would like to mention. Gilberto Calvillo and I have developed good direct algorithms for the optimum branching system problem, which might have some down to earth interest. Given a directed graph G, a value $c(j)$ and a capacity $d(j)$ for each edge, find a family of k branchings which together do not exceed the capacity of any edge and which together maximize total value. A branching in G is a forest such that each node of G has at most one edge of the forest directed toward it. Of course there are a number of equivalent problems but this one is convenient to say and to treat. By looking at the study of branchings and the study of optimum network flow in chapters of combinatorial optimization you might agree that the optimum branching systems problem is a natural gap. The analogous problem for forest systems in an undirected graph is solved by the matroid partitioning algorithm here together with the matroid greedy algorithm. The optimum branching system problem is quite different. It is solved in principle by a stew of matroid ideas including the ones here, and was first done that way, but it is better treated directly.

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Jack Edmonds

Matroid Partition

1. **Introduction.** Matroids can be regarded as a certain abstraction of matrices. They represent properties of matrices which are invariant under elementary row operations, namely properties of dependence among the columns. For any matrix over any field, there is a matroid whose elements correspond to the columns of the matrix and whose independent sets of elements correspond to the linearly independent sets of columns. A matroid M is completely determined by its elements and its independent sets of elements.

There are matroids which do not arise from any matrix over any field, so matroid theory does truly generalize an aspect of matrices. However, matroid theory is justified by new problems in matrix theory itself, in fact by problems in the special matrix theory of graphs (networks). It happens that an axiomatic matroid setting is natural for viewing these problems and that matrix machinery is superfluous for viewing them.

Much of matroid theory has been motivated by graphs. A graph G may be regarded as a matrix $N(G)$ of zeroes and ones, mod 2, which has exactly two ones in each column. The columns are the edges of the graph and the rows are the nodes of the graph. An edge and a node are said to meet if there is a one located in that column and that row. Of course a graph can also be regarded visually as a geometric network. It is often helpful to visualize statements on matroids for the case of graphs, though it can be misleading. Matroids do not contain objects corresponding to nodes or rows.

Another motivation here will be another source of matroids which is an extensive theory in its own right. It is well known in various contexts, including systems of distinct representatives, (0, 1)-matrices, network flows, matchings in graphs, and marriages. We will refer to it here as transversal theory.

2. **Problem.** The following definition of matroid has certain intrinsic interest.

A *matroid*, $M = (E, F)$, is a finite set E of elements and a non-empty family F of subsets of E, called *independent sets*, such that (1) every subset of an independent set is independent; and (2) for every set $A \subset E$, all maximal independent subsets of A have the same cardinality, called the *rank* $r(A)$ of A.

Any finite collection of elements and nonempty family of so-called independent sets of these elements which satisfies axiom 1 we shall call an *independence system*. This also happens to be the definition of an abstract simplicial complex, though the topology of complexes will not concern us.

It is easy to describe implicitly large independence systems which are apparently very unwieldy to analyze. For example, given a graph G, define an independent set of nodes in G to be such that no edge of G meets two nodes of the set.

The minimum coloring problem for an independence system is to find a partition of its elements into as few independent sets as possible.

A problem closely related to minimum coloring is the "packing problem". That is to find a maximum cardinality independent set. More generally the "weighted packing problem" is, where each element of the system carries a real numerical weight, to find an independent set whose weight-sum is maximum.

For any independence system, any *subsystem* consisting of a subset A of the elements and all of the independent sets contained in A is an independence system. Thus, a matroid is an independence system where the packing problem is postulated to be trivial for the system and all of its subsystems. After having spent much labor on packing problems, it is pleasant to study such systems. Matroids have a surprising richness of structure, as even the special case of graphic matroids shows.

A main result of this paper is a solution of the minimum coloring problem for the independent sets of a matroid. Another paper will treat the weighted packing problem for matroids.

3. **Ground rules.** One is tempted to surmise that a minimum coloring can be effected for a system by some simple process like extracting a maximal independent set to take on the first color, then extracting a maximal independent set of what is left to take on the second color, and so on till all elements are colored. This is usually far from being successful even for matroids.

Consider the class of matroids implicit in the class Π of all matrices over fields of integers modulo primes. (For large enough prime, this class includes the matroid of any matrix over the rational field.) We seek a good algorithm for partitioning the columns (elements of the matroid) of any one of the matrices (matroids) into as few sets as possible so that each set is independent. Of course, by carrying out the monotonic coloring procedure described above in all possible ways for a given matrix, one can be assured of encountering such a partition for the matrix, but this would entail a horrendous amount of work. We seek an algorithm for which the work involved increases only algebraically with the size of the matrix to which it is applied, where we regard the size of a matrix as increasing only linearly with the number of columns, the number of rows, and the characteristic of the field. As in most combinatorial problems, finding a finite algorithm is trivial but finding an algorithm which meets this condition for practical feasibility is not trivial.

We seek a good characterization of the minimum number of independent sets into which the columns of a matrix of Π can be partitioned. As the criterion of "good" for the characterization we apply the "principle of the absolute supervisor." The good characterization will describe certain information about the matrix which the supervisor can require his assistant to search out along with a minimum partition and which the supervisor can then use "with ease" to verify with mathematical certainty that the partition is indeed minimum. Having a good characterization does not mean necessarily that there is a good algorithm. The assistant might have to kill himself with work to find the information and the partition.

Theorem 1 on partitioning matroids provides the good characterization in the case of matrices of Π. The proof of the theorem provides a good algorithm in the case of matrices of Π. (We will not elaborate on how.) The theorem and the algorithm apply as well to all matroids via the matroid axioms. However, the "goodness" depends on having a good algorithm for recognizing independence.

4. **Theorem.** Let $\{M_i\}$, $i = 1, \cdots, k$, be an indexed family of matroids, $M_i = (E, F_i)$, all defined on the same set E of elements. Let $r_i(A)$ denote the rank of $A \subset E$ relative to M_i. Let $|A|$ denote the cardinality of A.

THEOREM 1. *Set E can be partitioned into a family $\{I_i\}$, $i = 1, \cdots, k$, of sets $I_i \in F_i$, if and only if there is no $A \subset E$ such that*

$$|A| > {}_i\sum r_i(A).$$

In particular, where the M_i's are the same matroid M, we have that: *The elements of a matroid M can be partitioned into as few as k sets, each independent in M, if and only if there is no set A of elements such that*

$$|A| > k \cdot r(A).$$

Proof of the "only if" part is easy. Suppose that $\{I_i\}$, $i = 1, \cdots, k$, is a partition of E such that I_i is independent in M_i. Then for any $A \subset E$,

$$|A| = {}_i\sum |A \cap I_i| \leqq {}_i\sum r_i(A).$$

5. **Lemmas.** A set $A \subset E$ is called *dependent* relative to a matroid $M = (E, F)$ if it is not a member of F.

Let A be any subset of the elements of a matroid M. Let I be any independent subset of A (relative to M). The set $S \subset A$, consisting of I and all elements $e \in A$ such that $I \cup e$ is dependent, is called the *span of I in A* (*with respect to M*).

LEMMA 1. *For any set A of the elements of a matroid M, and any independent set $I \subset A$, the span of I in A is the unique maximal set S such that $I \subset S \subset A$ and $r(S) = |I|$.*

PROOF. Let S be any maximal set such that $I \subset S \subset A$ and $r(S) = |I|$. Consider any $e \in A - I$. By the definition of rank, I is a maximal independent subset of S. Thus, if $e \cup I$ is independent, then $e \notin S$. And thus, on the other hand, if $e \cup I$ is dependent, then I is a maximal independent subset of $e \cup S$. Hence, in this latter case, by matroid-axiom 2, $r(e \cup S) = |I|$ and so $e \in S$. Thus, S is the span of I in A, and the lemma is proved.

Where I is any independent set of matroid M, not necessarily contained in set A, we will denote the span of $I \cap A$ in A, relative to matroid M, by $T(I, A, M)$.

A minimal dependent set of elements of a matroid M is called a *circuit* of M.

LEMMA 2. *The union of any independent set I and any element e of a matroid M contains at most one circuit of M.*

PROOF. Suppose $I \cup e$ contains two distinct circuits C_1 and C_2. Assume I is minimal for this possibility. We have $e \in C_1 \cap C_2$. There is an element $e_1 \in C_1 - C_2$ and an element $e_2 \in C_2 - C_1$. Set $(I \cup e) - (e_1 \cup e_2)$ is independent since otherwise $I - e_1$ is a smaller independent set than I for which $(I - e_1) \cup e$ contains more than one circuit. Set I and set $(I \cup e) - (e_1 \cup e_2)$ are maximal independent subsets of set $I \cup e$. This contradicts axiom 2.

6. **Algorithm.** Let $\{I_i\}$ $(i = 1, \cdots, k)$ be a family of mutually disjoint subsets of E such that I_i is independent in matroid $M_i = (E, F_i)$. Any number of these may be empty. Denote their union by $H = \cup(\{I_i\})$. Set H is said to be *partitionable* (relative to $\{M_i\}$).

Suppose there is an $e \in E - H$. We shall show how either to find an $A \subset H \cup e$ such that $|A| > {}_i\sum r_i(A)$, or else partition $H \cup e$, i.e., rearrange elements among the sets I_i to make room for e in one of them while preserving their mutual disjointness and their independence, respectively, in the matroids, M_i. This will prove the theorem.

This algorithm uses as a "primitive operation" the following: for any given index i, for any given set $I \subset E$ which is known to be independent in M_i, and for any element $e \in E - I$, determine that $I \cup e$ is independent in M_i or else find the $C \subset I \cup e$ such that C is a circuit of M_i. It is easy to see how this operation can be reduced to operations of the following type: determine whether or not $I \cup e$ is independent in M_i. In view of axiom 2, it is easy to see how, using either one of these types of operation, to determine $r_i(S)$ for any $S \subset E$.

PHASE 1 OF THE ALGORITHM. Let $S_0 = E$. For each $j - 1$, starting with $j - 1 = 0$, see if there is some i, call it $i(j)$, such that

$$|I_{i(j)} \cap S_{j-1}| < r_{i(j)}(S_{j-1}).$$

If so, let

$$S_j = T(I_{i(j)}; S_{j-1}; M_{i(j)})$$

be the span in S_{j-1}, with respect to matroid $M_{i(j)}$, of $I_{i(j)} \cap S_{j-1}$. Then repeat the above with $j - 1$ one greater. In this way, we construct a sequence $(I_{i(1)}, S_1), \cdots, (I_{i(n)}, S_n)$. The labels $i(j)$ are not necessarily distinct.

Set S_j is a proper subset of S_{j-1} since, by Lemma 1,

$$r_{i(j)}(S_j) = |I_{i(j)} \cap S_{j-1}| < r_{i(j)}(S_{j-1}).$$

Therefore, in order for the sequence to stop, we must reach an S_n such that for every i, $|I_i \cap S_n| = r_i(S_n)$. Because the I_i's are disjoint, this is equivalent to

$$|H \cap S_n| = {}_i\sum r_i(S_n).$$

Now, if $e \in S_n - H$, then where

$$A = (H \cap S_n) \cup e \subset S_n,$$

we have

$$|A| = |H \cap S_n| + 1 > {}_i\sum r_i(S_n) \geqq {}_i\sum r_i(A).$$

Therefore, according to the "only if" part of Theorem 1, since $A \subset H \cup e$ and $|A| > {}_i\sum r_i(A)$, set $H \cup e$ can not be partitioned.

On the other hand, where $e \in E - (H \cup S_n)$, we shall show how $H \cup e$ can be partitioned.

PHASE 2. Since $e \notin S_n$ and $e \in S_0$, and since the S_j's are nested, there is some S_h such that $e \notin S_h$ and such that $e \in S_j$ for $0 \leqq j < h$.

If $e \cup I_{i(h)}$ is independent, in $M_{i(h)}$, then adjoin e to $I_{i(h)}$ and we are done. Otherwise, let C be the circuit of matroid $M_{i(h)}$ which is contained in $e \cup I_{i(h)}$.

Set C is not contained in S_{h-1} because then, by the definition of the span function T and the construction of S_h, we would have $e \in S_h$. Let m be the smallest integer, $0 < m < h$, such that C is not contained in S_m.

Let e' be some member of $C - S_m$. By Lemma 2, $I'_{i(h)} = e \cup I_{i(h)} - e'$ is independent in $M_{i(h)}$. Replacing $I_{i(h)}$ by $I'_{i(h)}$, and letting $I'_i = I_i$ for $i \neq i(h)$, we now need to dispose of e' instead of e.

We know that $e' \notin S_m$ and that $e' \in S_j$ for $0 \leqq j < m$. We can also show that sequence, $(I'_{i(1)}, S_1), \cdots, (I'_{i(m)}, S_m)$, is of the same construction as sequence, $(I_{i(1)}, S_1), \cdots, (I_{i(h)}, S_h)$:

Consider the terms, $j = 1, \cdots, m$, in order. If $i(j) = i(h)$, then $I'_{i(j)} = e \cup I_{i(j)} - e'$. Since $C \subset S_{j-1}$, the set $D = (I'_{i(j)} \cup e') \cap S_{j-1} = (I_{i(j)} \cup e) \cap S_{j-1}$ is dependent in $M_{i(j)}$, and thus

$$r_{i(j)}(D) = |I'_{i(j)} \cap S_{j-1}| = |I_{i(j)} \cap S_{j-1}|.$$

Therefore by the uniqueness asserted in Lemma 1 we have that

$$T(I'_{i(j)}, S_{j-1}; M_{i(j)}) = T(I_{i(j)}; S_{j-1}; M_{i(j)}) = S_j.$$

This relation obviously also holds if $i(j) \neq i(h)$, since then $I'_{i(j)}$

$= I_{i(j)}$. Thus we can treat e' in the same manner that we treated e, and so on. Since m is a positive integer strictly less than h, we will be done with this application of Phase 2 in less than h iterations. That completes the description of the algorithm and the proof of Theorem 1.

7. **Some applications.** For any integer t, $0 \leqq t \leqq |E|$, let F^t_* be the family of subsets of E which have cardinality at most t. Clearly, $M^t_* = (E, F^t_*)$ is a matroid, and the rank function r^t_* of M^t_* is $r^t_*(A) = \min(t, |A|)$.

APPLICATION 1. For any indexed family $\{M_i\}$ of matroids, $M_i = (E, F_i)$, $i = 1, \cdots, k$, consider Theorem 1 applied to the indexed family consisting of $\{M_i\}$ together with matroid M^t_*. Clearly, set E can be partitioned into sets which are independent respectively in matroid M^t_* and matroids M_i if and only if at least some $|E| - t$ members of E can be partitioned into sets which are independent respectively in matroids M_i. Thus Theorem 1 says that the latter can be done if and only if there is no $A \subset E$ such that

$$|A| > r^t_*(A) + {}_i\sum r_i(A), \quad \text{i.e., } |A| > t + {}_i\sum r_i(A).$$

APPLICATION 2. In particular where $|E| - t = {}_i\sum r_i(E)$, we have that: there exist mutually disjoint subsets of E which are bases (maximum cardinality independent sets) respectively of matroids M_i if and only if there is no $A \subset E$ such that $|A| > |E| - \sum r_i(E) + \sum r_i(A)$, i.e., such that $|E - A| < {}_i\sum (r_i(E) - r_i(A))$.

Where the M_i's are the matroids of graphs, this result is equivalent, using rank-properties of graphs, to a theorem of Tutte [**11**], and also of Nash-Williams [**7**] where the graphs are identical.

Similarly, Theorem 1 itself implies the following theorem of Nash-Williams [**8**].

The edges of a graph G can be partitioned into as few as k forests if and only if there is no subset U of nodes in G such that, where E_U is the set of edges in G which have both ends in U,

$$|E_U| > k(|U| - 1).$$

Nash-Williams' theorem follows (we omit the proof) from Theorem 1 by using the following characterization of the rank function of a graph due to Whitney:

The rank $r(A)$ of any subset A of edges in G, i.e., the rank of the matroid subset corresponding to A, equals the number of nodes minus the number of connected components in the subgraph consisting of the edges A and the nodes they meet, or equivalently in the subgraph consisting of the edges A and all the nodes in G.

For the case where the M_i's are identical sets of vectors in a vector space (with respect to linear independence), Theorem 1 is proved by Horn [5] and Rado [9]. In fact, Rado posed the problem of deciding whether or not the result extends to matroids. I did not know of their work until the present work was completed.

For any integer $t \geqq 0$, and any matroid $M = (E, F)$, let F^t consist of the members of F which have cardinality at most t. Clearly, $M^t = (E, F^t)$ is a matroid, called the *truncation* of M at t.

APPLICATION 3. For any matroid M and for any prescribed integers $t(i)$, $i = 1, \cdots, k$, such that $0 \leqq t(i) \leqq r(E)$, let $M_i = M^{t(i)}$. Applying Theorem 1 to this family of matroids gives n. and s. conditions for there to be a family of independent sets of M of specified sizes $t(i)$, $i = 1, \cdots, k$, whose union is E.

APPLICATION 4. Applying the result of Application 2, to this same family of matroids gives n. and s. conditions for there to be a family of mutually disjoint independent sets of M having specified sizes $t(i)$.

For any matroid $M = (E, F)$ and any prescribed independent set $J \in F$, let F^J consist of sets I such that $J \cap I = \emptyset$ and such that $(J \cup I) \in F$. Clearly, $M^J = (E, F^J)$ is a matroid.

APPLICATION 5. For any matroid $M = (E, F)$ and any prescribed family of mutually disjoint independent sets $J(i) \in F$, $i = 1, \cdots, k$, let $M_i = M^{J(i)}$. Applying Theorem 1 to this family of matroids gives n. and s. conditions for there to be a partition of E into independent sets $I_i \in F$ such that $J_i \subset I_i$.

APPLICATION 6. Applying the result of Application 2 to this same family of matroids gives n. and s. conditions for there to be a family of mutually disjoint bases I_i of M such that $J_i \subset I_i$.

For any matroid $M = (E, F)$ and any $A \subset E$, let F^A consist of the sets $I \in F$ such that $I \subset A$. Clearly $M^A = (E, F^A)$ is a matroid.

For any matroid $M = (E, F)$ and any prescribed family of sets $A(i) \subset E$, $i = 1, \cdots, k$, let $M_i = M^{A(i)}$ and so on.

One can combine these and later constructions in other ways. The algorithm of course applies as well as the theorem.

8. **A class of Abelian semigroups.** Relative to any indexed family $\{M_i\}, i = 1, \cdots, k$, of independence systems, $M_i = (E, F_i)$, on the same set E, a set $H \subset E$ is called *partitionable* if it can be expressed in the form $H = I_1 \cup \cdots \cup I_k$ where $I_i \in F_i$. We may, of course, without further restricting H, require that these I_i's be mutually

disjoint. Let F denote the family of subsets H of E which are partitionable relative to $\{M_i\}$. Clearly, $M = (E, F)$ is an independence system. It is denoted as $M = {}_i\sum M_i$ and called the *sum* of the systems M_i.

It is easy to show that this sum is associative, and commutative, and has a unique identity element. Thus, the independence systems on a given set E form an abelian semigroup, say G, under this operation. The theorem below shows that all the matroids on set E form a sub-semigroup of G.

THEOREM 2. *Where the M_i's are matroids, ${}_i\sum M_i$ is a matroid.*

We have only to prove that this system satisfies matroid-axiom 2.

Let H be any member of F. Letting this H be the H of the matroid-partition algorithm, we get a certain set $S_n \subset E$ such that

$$|H \cap S_n| = {}_i\sum r_i(S_n).$$

We showed during Phase 1 of the algorithm that, for any $e \in S_n - H$, the set $H \cup e$ is not partitionable. Thus H is a maximal partitionable subset of $H \cup S_n$. We must show the stronger fact that H is a maximum-cardinality partitionable subset of $H \cup S_n$. That is, for any partitionable subset H' of $H \cup S_n$, $|H'| \leqq |H|$.

Since $H' \subset H \cup S_n$, we have

$$|H' - S_n| \leqq |H - S_n|.$$

Since H' is partitionable, we have, using the "only if" part of Theorem 1,

$$|H' \cap S_n| \leqq {}_i\sum r_i(H' \cap S_n) \leqq {}_i\sum r_i(S_n) = |H \cap S_n|.$$

Adding these two inequalities together, we get $|H'| \leqq |H|$.

Let A be any subset of E. Suppose that the above H is any maximal partitionable subset of A. Phase 2 of the partitioning algorithm shows that, for any $e \in E - (H \cap S_n)$, the set $H \cup e$ is partitionable. Therefore, $A \subset H \cup S_n$. Therefore, H is a maximum-cardinality partitionable subset of A, as well as of $H \cup S_n$. Thus, system M satisfies matroid-axiom 2, and so Theorem 2 is proved.

(There are at least two other proofs of Theorem 2 which do not use the partition-algorithm. They appear respectively in the as yet unpublished papers *Matroids and the greedy algorithm* and *Submodular set functions*. The latter paper describes a more general construction, from which the present matroid-sums, as well as the "transversal matroids" which we are about to describe, were originally derived.)

"Matroid-sums" is a useful "nonmatric" way to construct matroids. Let M_1 and M_2 be matroids determined by matrices N_1 and N_2 where the columns of both N_1 and N_2 are indexed by the set E. It can be shown that if N_1 and N_2 are matrices over different fields, say the integers modulo different primes, then matroid $M_1 + M_2$ is not generally the matroid of any matrix over any field.

It can be shown using an extension of the technique in [**2**], that if N_1 and N_2 are matrices over the same commutative field, Ψ, then $M_1 + M_2$ is the matroid of a matrix, say N, over a field extension of Ψ by about $|E|$ indeterminates. However, as indicated in [**2**], even in this case one would not determine whether a set is indepe-pendent in $M_1 + M_2$ by pivoting in N, but rather by applying the matroid-partition algorithm to M_1 and M_2.

9. **Transversal matroids.** An interesting special case of matroid-sum, ${}_i\sum M_i$, is where all the summands are rank-one matroids. Clearly, $r_i(E) = 1$ if and only if every member of F_i, besides the empty set, consists of a single element of E.

Assume $r_i(E) = 1$, $i = 1, \cdots, k$. Let Q_i consist of the elements of E which are not themselves dependent in M_i. A set $H \subset E$ which is partitionable relative to $\{M_i\}$ is called a partial transversal, or a partial *SDR*, of the indexed family $\{Q_i\}$. Matroid $M = {}_i\sum M_i$, for this case, is called a *transversal matroid*.

Clearly, the partial transversals of any indexed family, $\{Q_i\}$, $i = 1, \cdots, k$, (of not-necessarily-distinct sets $Q_i \subset E$) are the sets $H \subset E$ which can be expressed in the form $H = I_1 \cup \cdots \cup I_k$ where I_i either consists of a single element of Q_i or else is empty. We may, of course, without further restricting H, require that these I_i's be mutually disjoint. Set H is called a *transversal* or an *SDR of* $\{Q_i\}$ if the I_i's are mutually disjoint and nonempty, i.e., if $|H| = k$. Thus a partial transversal of family $\{Q_i\}$ is a set which is a trans-versal of some subfamily of $\{Q_i\}$.

Theorem 2 shows that the partial transversals of a $\{Q_i\}$ are the independent sets of a matroid $M = (E, F)$, a transversal matroid. Given $\{Q_i\}$, the partition algorithm provides a good algorithm for deciding whether or not any given subset of E is independent in M, for computing the rank of M, and so on.

Matroid theory and transversal theory enhance each other via transversal matroids, as do matroid theory and graph theory via graphic matroids.

P. J. Higgins [**4**] gives *n.* and *s.* conditions for an indexed family

$\{Q_i\}$, $i = 1, \cdots, k_0$, of sets to have k_1 mutually disjoint partial transversals of prescribed sizes $t(1), t(2), \cdots, t(k_1)$. This is the subject of Application 4 in §7 by taking the M there to be the transversal matroid of $\{Q_i\}$.

Much of the present article consists of extractions from [**1**] and [**3**]. Paper [**3**] contains another view of transversal matroids, and also an alternative approach to the partition of transversal matroids using the max-flow min-cut theorem and the integrity theorem of Ford and Fulkerson.

For other related material see [**6**], [**10**], [**12**], the preceding article, and the next one.

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